Lecture 22

Properties of ACF's

2. $R_X(\tau) = R_X(-\tau) \Rightarrow$ ACF is an even function

Why?

$$R_X(\tau) = E[X(t)\cdot X(t+\tau)] = E[\underbrace{X(t+\tau)\cdot X(t)}_{t-(t+\tau)=-\tau}] = R_X(-\tau)$$

3. $R_X(\tau) \leq R_X(0)$

From Property 1:

$$R_X(0) = E[X^2(t)] = E[X^2]$$

Proof:

$$(X_1 - X_2)^2 \geq 0$$

$$\Rightarrow E[(X_1 - X_2)^2] \geq 0 \Rightarrow E[X_1^2 - 2X_1X_2 + X_2^2] \geq 0$$

$$\Rightarrow E[X_1^2] - 2E[X_1X_2] + E[X_2^2] \geq 0$$

$$\Rightarrow \underbrace{E[X_1^2]}_{R_X(0)} + \underbrace{E[X_2^2]}_{R_X(0)} \geq 2\underbrace{E[X_1X_2]}_{2\cdot R_X(\tau)} = 2E[X(t)\cdot X(t+\tau)]$$

$$\Rightarrow 2R_X(0) \geq 2R_X(\tau)$$

4. Constant Component

If $X(t)$ has a non-zero "DC" component, then $R_X(\tau)$ has a constant component.

For example, if $X(t) = A$,

$$R_X(\tau) = E[X(t)\cdot X(t+\tau)] = E[A\cdot A] = A^2$$

Another example: $X(t) = \underset{\substack{\uparrow \\ \text{constant} \neq 0}}{\bar{X}} + N(t)$

where $E[N(t)] = E[N(t+\tau)] = 0$

Thus, $E[X(t)] = E[\bar{X} + N(t)] = E[\bar{X}] + E[N(t)] = \bar{X}$

Question: $R_X(\tau)$?

Answer:

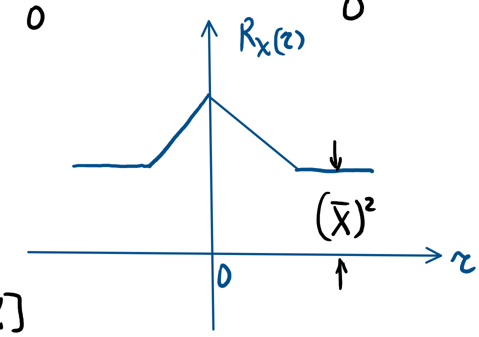
$$R_X(\tau) = E[X(t) \cdot X(t+\tau)]$$

$$= E\{[\bar{X} + N(t)] \cdot [\bar{X} + N(t+\tau)]\}$$

$$= E[\bar{X} \cdot \bar{X}] + E[\bar{X} \cdot N(t+\tau)] + E[N(t)\bar{X}] + \underbrace{E[N(t) \cdot N(t+\tau)]}$$

$$= E[(\bar{X})^2] + \bar{X} \cdot \underbrace{E[N(t+\tau)]}_{0} + \bar{X} \cdot \underbrace{E[N(t)]}_{0} + \underbrace{R_N(t+\tau-t)}$$

$$R_X(\tau) = (\bar{X})^2 + R_N(\tau)$$



$\lim\limits_{\tau \to \infty} R_X(\tau) = (\bar{X})^2 \neq 0$

$R_X(\tau)$ has a constant component.

$\sqrt{(\bar{X})^2} = \pm \bar{X} = \pm E[X]$

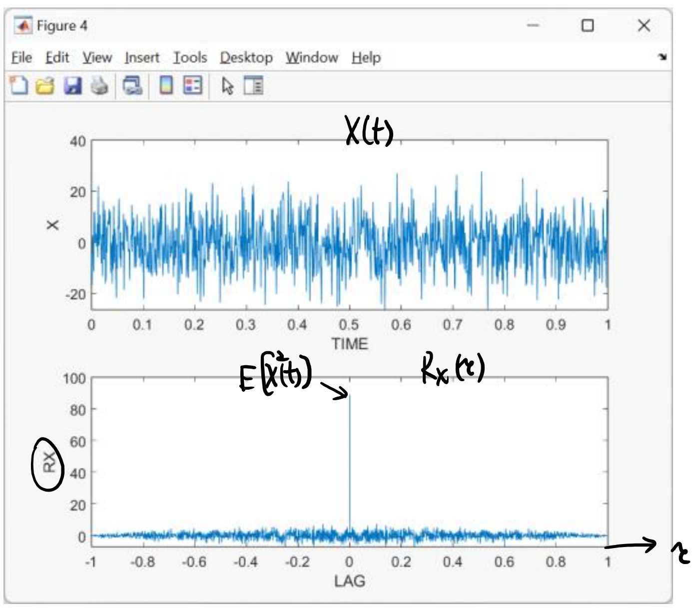
ambiguity

$X(t)$ has a non-zero "DC" component

```
%Example of autocorrelation calculation
rand('seed', 1000); % use seed to make repeatable
x=10*randn(1, 1001); % generate random samples
t = 0: 0.001 : 1; % sampling index
[t1, R] = corb(x, x, 1000) ; % autocorrelation
subplot(2, 1, 1);
plot(t,x);
xlabel('TIME'); ylabel('X');
subplot(2, 1, 2);
plot(t1,R);
xlabel('LAG'); ylabel('RX');
```

```
function [ndt, R] = corb(a, b, f)
%   Correlation functing using convolution
N = length(a);
R = conv(a, fliplr(b))/(N + 1);
ndt = (-(N - 1): (N - 1))/f;
end
```

http://www.ece.uah.edu/~dwpan/course/ee385/code/corrx_example1.m
http://www.ece.uah.edu/~dwpan/course/ee385/code/corb.m



- Property 5 of ACF's

If $X(t)$ has a periodic component, then $R_X(\tau)$ also has a periodic component.

For example, $X(t) = A\cos(\omega t + \theta)$

A, ω: constants; θ: RV

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{elsewhere} \end{cases}$$

Question: $R_X(\tau) = ?$

$$R_X(\tau) = E[X(t)\cdot X(t+\tau)]$$

$$= E[A\cos(\omega t+\theta)\cdot A\cos(\omega t+\omega\tau+\theta)]$$

$$= E[A^2\cos(\omega t+\theta)\cos(\omega t+\omega\tau+\theta)]$$

$$= A^2 E[\cos(\omega t+\theta)\cdot\cos(\omega t+\omega\tau+\theta)]$$

Use Table A-1

$$\cos(\alpha)\cos(\beta) = \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$R_X(\tau) = \frac{A^2}{2} E\left[\cos\left(2\omega t+\omega\tau+2\theta\right)+\cos(\omega\tau)\right]$$

$$= \frac{A^2}{2} E[\cos(2\omega t+\omega\tau+2\theta)] + \frac{A^2}{2}\underbrace{E[\cos(\omega\tau)]}_{\cos(\omega\tau)}$$

$$\int_0^{2\pi} \underset{\substack{\uparrow \\ \text{PDF}}}{\frac{1}{2\pi}}\cdot\cos(2\omega t+\omega\tau+2\theta)\,d\theta$$

$$= \frac{1}{2\pi}\sin\left(2\omega t+\omega\tau+2\theta\right)/2\,\Big|_0^{2\pi} = 0$$

In Summary,

$$R_X(\tau) = \frac{A^2}{2}\cos(\omega\tau)$$