Lecture 23

- ACF of a Sum Process

$$Z(t) = X(t) + Y(t)$$

Question: $R_Z(\tau)$?

Answer:

$$R_Z(\tau) = E\left[Z(t)\, Z(t+\tau)\right]$$

$$= E\left\{\left[X(t) + Y(t)\right] \cdot \left[X(t+\tau) + Y(t+\tau)\right]\right\}$$

$$= \underbrace{E\left[X(t)\, X(t+\tau)\right]}_{R_X(\tau)} + \underbrace{E\left[X(t)\, Y(t+\tau)\right]}_{?} + \underbrace{E\left[Y(t)\, X(t+\tau)\right]}_{?} + \underbrace{E\left[Y(t) \cdot Y(t+\tau)\right]}_{R_Y(\tau)}$$

$$= R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau)$$

where

$$R_{XY}(\tau) = E\left[X(t)\, Y(t+\tau)\right]$$

$$R_{YX}(\tau) = E\left[Y(t)\, X(t+\tau)\right]$$

- Cross-Correlation Functions

In general, $R_{XY}(\tau) \neq R_{YX}(\tau)$

Exceptions:

(1) If $\tau=0$, $R_{XY}(0) = E[X(t)\cdot Y(t+0)] = E[Y(t)\cdot X(t)] = R_{YX}(0)$

(2) If $X(t)$ and $Y(t)$ are mutually independent

$$R_{XY}(\tau) = E[X(t)\cdot Y(t+\tau)] = E[X(t)]\cdot E[Y(t+\tau)] = \bar{X}\cdot\bar{Y}$$

$$R_{YX}(\tau) = E[Y(t)\cdot X(t+\tau)] = E[Y(t)]\cdot E[X(t+\tau)] = \bar{Y}\cdot\bar{X}$$

Thus $R_{XY}(\tau) = R_{YX}(\tau)$

And if $X(t)$ and $Y(t)$ are independent,

$$Z(t) = X(t) + Y(t)$$

$$R_Z(\tau) = R_X(\tau) + \underbrace{R_{XY}(\tau)} + \underbrace{R_{YX}(\tau)} + R_Y(\tau)$$

$$= R_X(\tau) + \bar{X}\cdot\bar{Y} + \bar{Y}\cdot\bar{X} + R_Y(\tau)$$

Furthermore, if at least one of $\bar{X}$ and $\bar{Y}$ is zero (zero mean)

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau)$$

- Cross Correlation Functions between a Sum and Difference Processes

$$\begin{cases} U(t) = X(t) + Y(t) \\ V(t) = X(t) - Y(t) \end{cases}$$

Question: $R_{UV}(\tau)$, and $R_{VU}(\tau)$?

$$R_{UV}(\tau) = E\left[U(t)\cdot V(t+\tau)\right]$$

$$= E\left\{\left[X(t)+Y(t)\right]\cdot\left[X(t+\tau)-Y(t+\tau)\right]\right\}$$

$$= \underbrace{E\left[X(t)X(t+\tau)\right]} - E\left[X(t)Y(t+\tau)\right] + E\left[Y(t)X(t+\tau)\right] - E\left[Y(t)Y(t+\tau)\right]$$

$$= R_X(\tau) - R_{XY}(\tau) + R_{YX}(\tau) - R_Y(\tau) \qquad \cdots\cdots (1)$$

$$R_{VU}(\tau) = E\left[V(t)\cdot U(t+\tau)\right]$$

$$= E\left\{\left[X(t)-Y(t)\right]\cdot\left[X(t+\tau)+Y(t+\tau)\right]\right\}$$

$$= E\left[X(t)X(t+\tau)\right] + E\left[X(t)Y(t+\tau)\right] - E\left[Y(t)X(t+\tau)\right] - E\left[Y(t)Y(t+\tau)\right]$$

$$= R_X(\tau) + R_{XY}(\tau) - R_{YX}(\tau) - R_Y(\tau) \qquad \cdots\cdots (2)$$

If X(t) and Y(t) are independent, and both X(t) and Y(t) have zero mean, then

$$R_{UV}(\tau) = R_{VU}(\tau) = R_X(\tau) - R_Y(\tau).$$

Previously, estimation of autocorrelation function.

```
%Example of autocorrelation calculation
rand('seed', 1000); % use seed to make repeatable
x=10*randn(1, 1001); % generate random samples
t = 0: 0.001 : 1; % sampling index
[t1, R] = corb(x, x, 1000) ; % autocorrelation
subplot(2, 1, 1);
plot(t,x);
xlabel('TIME'); ylabel('X');
subplot(2, 1, 2);
plot(t1,R);
xlabel('LAG'); ylabel('RX');
function [ndt, R] = corb(a, b, f)
%   Correlation functing using convolution
N = length(a);
R = conv(a, fliplr(b))/(N + 1);
ndt = (-(N - 1): (N - 1))/f;
end
```

```
function [ndt, R] = corb(a, b, f)
%   Correlation function using convolution
N = length(a);
R = conv(a, fliplr(b))/(N + 1);
ndt = (-(N - 1): (N - 1))/f;
end
```

fliplr

Flip array left to right

```
>> x = 1: 5
x =
    1    2    3    4    5

>> fliplr(x)
ans =
    5    4    3    2    1
```

In general,

Correlation: $$z[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n+k]$$

Covolution: (Sum) $$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

Use Convolution to implement Correlation by first flipping $h[n]$.

http://www.ece.uah.edu/~dwpan/course/ee385/code/ccf_example.m

```
% Example of CCF calculation to identify signal from noise

clear all;
noise = 3*randn(1, 1001); % noise
theta = random('Uniform', 0, 2*pi);
t = 0: 0.001 : 1; % sampling index
x = cos(100*pi*t + theta);  % random signal
y = x + noise;
[t1, Ry] = corb(y, y, 1000) ; % autocorrelation
xd = cos(100*pi*t);  % deterministic signal for detection
[t2, Rxy] = corb(xd, y, 1000) ; % cross-correlation
subplot(3, 1, 1);
plot(t,y);
xlabel('TIME'); ylabel('Y');
subplot(3, 1, 2);
plot(t1,Ry);
xlabel('LAG'); ylabel('R_Y');
subplot(3, 1, 3);
plot(t2,Rxy);
xlabel('LAG'); ylabel('R_{XY}');
```