Lecture 24

Chapter 7: Spectral Density

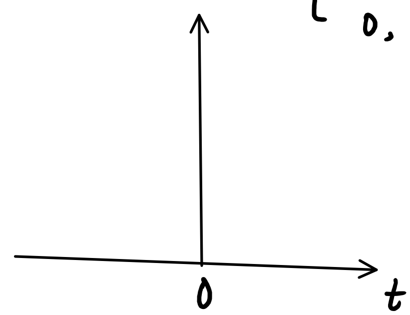
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

impulse response — Convolution integral

If $x(t) = \delta(t)$, then $y(t) = h(t)$.

$$\delta(t) = \begin{cases} \infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



Fourier Transform (FT)

$x(t) \xrightarrow{FT} X(\omega)$, where

$$\begin{cases} X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\,dt & \text{Forward Transform} \\ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\,d\omega & \text{Inverse Transform} \end{cases}$$

$h(t) \xrightarrow{FT} H(\omega)$, $\quad y(t) \xrightarrow{FT} Y(\omega)$

Convolution Integration $\xrightarrow{FT}$ Multiplication

$$y(t) = x(t) * h(t) \longleftrightarrow Y(\omega) = X(\omega) \cdot H(\omega)$$

$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$

$X(t)$: Random Process (W.S.S.)

$h(t)$: impulse response

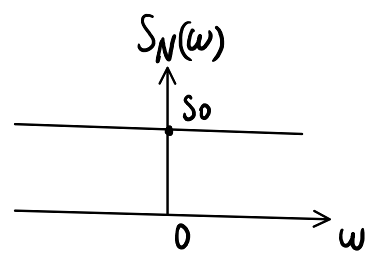
$Y(t)$: Random Process (W.S.S.)

$R_X(\tau)$: Autocorrelation is given

$\downarrow \mathcal{FT}$

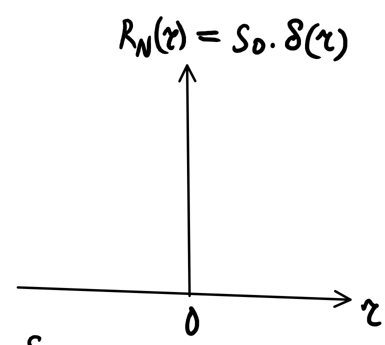
$$\underbrace{S_X(\omega)}_{\text{Spectral Density}} = \mathcal{FT}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j\omega\tau}\, d\tau$$

- White Noise
  $\underbrace{\text{Random Process}}_{N(t)}$ that has a constant value: S_0



for its spectral density: $S_N(\omega) = S_0$.

Question : $R_N(\tau)$?



Answer : If $R_N(\tau) = S_0 \cdot \delta(\tau)$,

then $S_N(\omega) = \mathcal{FT}\{R_N(\tau)\}$

$$= \int_{-\infty}^{\infty} S_0 \cdot \delta(\tau) \cdot e^{-j\omega\tau}\, d\tau = S_0.$$

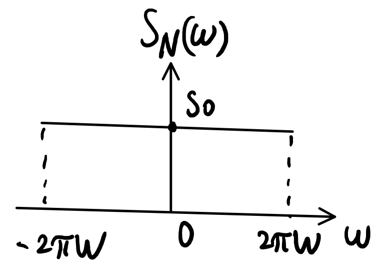
```
>> Z = randn(1, 73113);    % Noise
>> plot(Z)
>> sound(Z, 8192)

>> load handel     % Signal
```

```
>> figure; plot(y)
>> sound(y)
```

Band-Limited White Noise

$$S_X(\omega) = \begin{cases} S_0, & -2\pi W \leq \omega \leq 2\pi W \\ 0, & \text{elsewhere} \end{cases}$$



Question: $R_X(\tau) = ?$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega \qquad \text{Inverse Transform}$$

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega) \cdot e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} S_0 \cdot e^{j\omega\tau}\, d\omega$$

$$= \frac{S_0}{2\pi} \cdot \left.\frac{e^{j\omega\tau}}{j\tau}\right|_{-2\pi W}^{2\pi W} = \frac{S_0}{2\pi}\,\frac{1}{j\tau} \cdot \left(e^{j\overset{\theta}{2\pi W\tau}} - e^{-j\overset{\theta}{2\pi W\tau}}\right)$$

where

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

Thus $$R_X(\tau) = \frac{S_0}{2\pi j\tau} \cdot 2j\sin(2\pi W\tau) = \frac{S_0 \sin(2\pi W\tau)}{\pi\tau}$$

Thus $$R_X(\tau) = 2S_0W\, \text{Sinc}(\underbrace{2W\tau}) = \cancel{2}S_0\cancel{W} \cdot \frac{\text{Sin}(2\pi W\tau)}{\cancel{2}\pi\cancel{W}\tau}$$

sinc

The sinc function is defined by

$$\text{sinc}(t) = \begin{cases} \dfrac{\sin \pi t}{\pi t} & t \neq 0, \\ 1 & t = 0. \end{cases}$$

if $t =$ integers $= k$ $(\neq 0)$

then

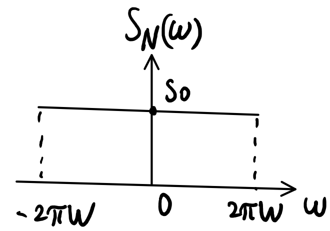
$$\text{Sinc}(t) = \frac{\sin(k\pi)}{k\pi} = 0$$

$1 \quad t = 0.$

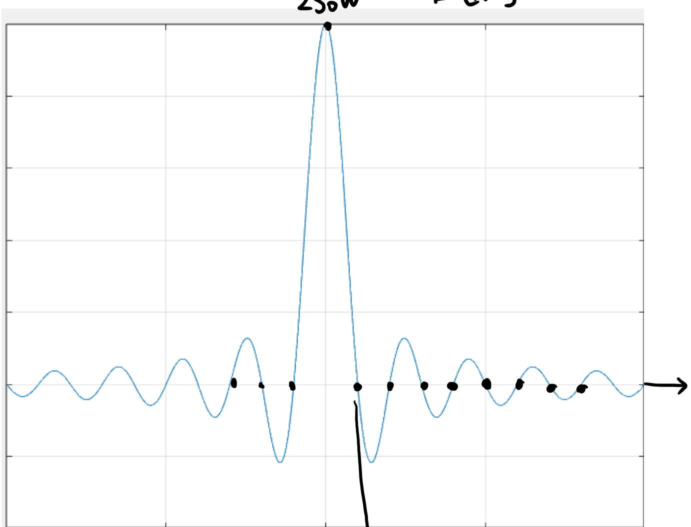
then

$\text{sinc}(t) = \frac{\sin(k\pi)}{k\pi} = 0$

$R_X(\tau) = 2S_0W \,\mathrm{sinc}(2W\tau) \xrightarrow{FT} S_X(\omega) =$



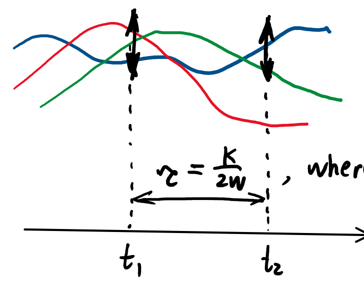
$2S_0W = E[X^2]$



$2W\tau = 1 \quad \left(R_X\left(\frac{1}{2W}\right) = 0 \right)$

$\Rightarrow \tau = \frac{1}{2W} \qquad$ As $W \uparrow \;\Rightarrow \tau = \frac{1}{2W} \downarrow$

$X(t) \rightarrow E[X(t_1)X(t_2)] = 0$



$\tau = \frac{k}{2W}$, where k is an integer $(\neq 0)$