Lecture 25

- Cross Spectral Densities (CSD)

$$\text{CCF} \underset{FT^{-1}}{\overset{FT}{\rightleftarrows}} \text{CSD}$$

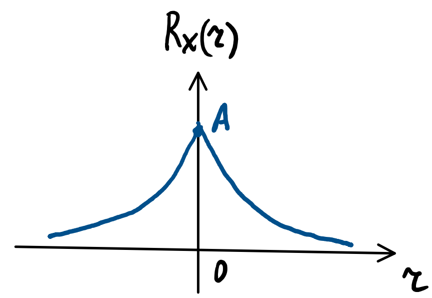
$$R_{XY}(\tau) = E\left[X(t)\,Y(t+\tau)\right] \xrightarrow{FT} S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{YX}(\tau) = E\left[Y(t)\,X(t+\tau)\right] \xrightarrow{FT} S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

Example: Given a ACF, determine the Spectral Density.

$$R_X(\tau) = A e^{-\beta|\tau|}, \quad A>0, \text{ and } \beta>0$$

Question: $S_X(\omega)$?



Answer:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0} A e^{-\beta(-\tau)}\, e^{-j\omega\tau}\, d\tau + \underbrace{\int_{0}^{\infty} A e^{-\beta\cdot\tau} \cdot e^{-j\omega\tau}\, d\tau}_{?}$$

where

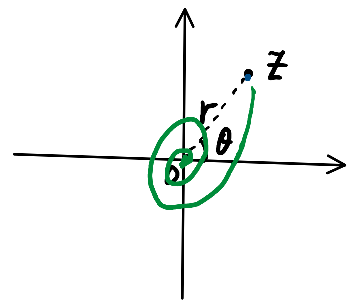
$$\int_0^{\infty} A\, e^{-(\beta+j\omega)\tau}\, d\tau = \left.\frac{A e^{-(\beta+j\omega)\tau}}{-(\beta+j\omega)}\right|_0^{\infty}$$

$$= \frac{A}{-(\beta+j\omega)}\left[e^{-(\beta+j\omega)\infty} - \underbrace{e^{-(\beta+j\omega)\cdot 0}}_{1}\right]$$

$$e^{-(\beta+j\omega)\infty} = \lim_{\tau\to\infty} e^{-(\beta+j\omega)\tau} = \lim_{\tau\to\infty} \underbrace{e^{-\beta\tau}}_{r} \cdot e^{\overbrace{-j\omega\tau}^{\theta = -\omega\tau}} = 0$$

Any complex number $z = re^{j\theta}$

$$\lim_{\tau\to\infty} e^{-\beta\tau} = \lim_{\tau\to\infty} \left(e^{-\beta}\right)^{\tau} \to 0$$



$$\int_0^\infty Ae^{-\beta\cdot\tau} \cdot e^{-j\omega\tau} d\tau = \frac{A}{\beta+j\omega}$$

Similarly,

$$\int_{-\infty}^0 Ae^{-\beta(-\tau)} e^{-j\omega\tau} d\tau = \int_{-\infty}^0 Ae^{(\beta-j\omega)\tau} d\tau$$

$$= \frac{A}{\beta-j\omega} \cdot e^{(\beta-j\omega)\tau}\Big|_{-\infty}^{0} = \frac{A}{\beta-j\omega}$$

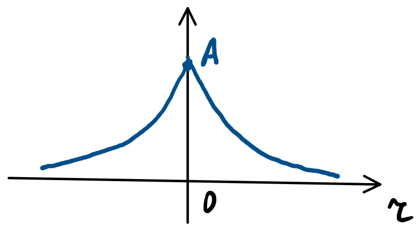
Hence,

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau$$

$$= \frac{A}{\beta-j\omega} + \frac{A}{\beta+j\omega} = \frac{A(\beta+j\omega+\beta-j\omega)}{(\beta-j\omega)(\beta+j\omega)} = \frac{2A\beta}{\beta^2+\omega^2}$$

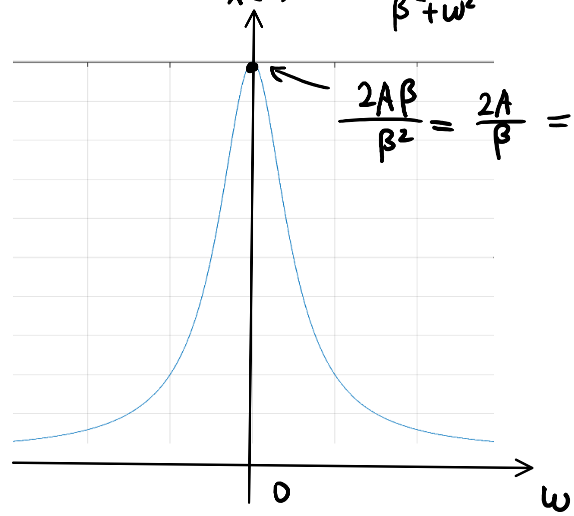
e.g., $A=\beta=1$, $S_X(\omega) = \frac{2}{1+\omega^2}$

$$R_X(\tau) = Ae^{-\beta|\tau|}$$



$$\xrightarrow{FT}$$

$$S_X(\omega) = \frac{2A\beta}{\beta^2+\omega^2}$$



$$\frac{2A\beta}{\beta^2} = \frac{2A}{\beta} = S_X(0)$$

$$\lim_{\tau\to\infty} R_X(\tau) = \lim_{\tau\to\infty} Ae^{-\beta|\tau|} = 0$$

Previously,

$R_X(\tau)$: Autocorrelation is given

$$\Big\downarrow FT$$

$$\underbrace{S_X(\omega)}_{\text{Spectral Density}} = FT\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$S_X(\omega) \xrightarrow{FT^{-1}} R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, e^{j\omega\tau}\, d\omega$$

$$\tau = 0 \Downarrow$$

$$R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, e^{j\omega 0}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$$

Since

$$R_X(0) = E[X^2(t)] = E[X^2]$$

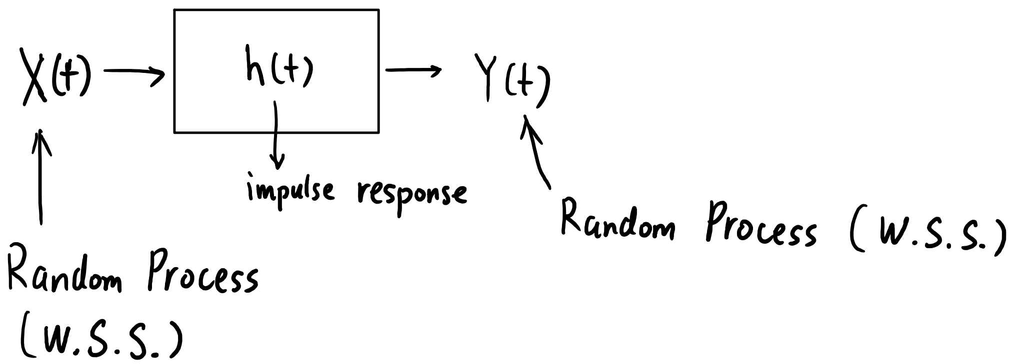
Thus

$$E[X^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$$

Everage Power (Power) Spectral Density

$$A = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2A\beta}{\beta^2+\omega^2}\,d\omega \quad ?$$

Chapter 8: Response of Linear Systems to Random Inputs



- Given the mean of the input, $\overline{X(t)} = \bar{X}$

  determine $\overline{Y(t)}$?

$$\overline{Y(t)} = E\left[Y(t)\right] = E\left\{\int_{-\infty}^{\infty} X(t-\lambda)\, h(\lambda)\, d\lambda\right\} = \int_{-\infty}^{\infty} E\left[X(t-\lambda)\, h(\lambda)\right] d\lambda$$

where $E\left[X(t-\lambda)\, h(\lambda)\right] = h(\lambda)\, E\left[X(t-\lambda)\right] = h(\lambda) \cdot \bar{X}$

Thus

$$\boxed{\bar{Y} = \bar{X} \cdot \int_{-\infty}^{\infty} h(\lambda)\, d\lambda}$$

Assume that the system is causal: $y(t)$ does not depend on $\underbrace{x(\lambda)}_{\text{future inputs}}$, where $\lambda > t$

$\boxed{\text{If } h(\lambda) = 0, \text{ for } \lambda < 0, \text{ then the system is causal.}}$

$$\int_{-\infty}^{\infty} \underbrace{X(t-\lambda)}_{t-\lambda > t}\, h(\lambda)\, d\lambda$$

$h(\lambda) = 0$ if $\lambda < 0$

$X(t-\lambda)$ where $t-\lambda > t$

Future inputs $\times\ 0 = 0$