Lecture 21

2-D DFT and its Properties

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

Translation:

$$f(x,y)e^{j2\pi\left(\frac{u_0 x}{M}+\frac{v_0 y}{N}\right)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi\left(\frac{x_0 u}{M}+\frac{y_0 v}{N}\right)}$$

Example: M = N = 2

```
>> f = [ 1 2; 3 4]
f =
   1   2
   3   4
```

$$= \begin{bmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{bmatrix}$$

$$F(0,0) = \sum_{x=0}^{1}\sum_{y=0}^{1} f(x,y) = 1+2+3+4 = 10$$

$$\underset{u\ \ v}{F(0,1)} = \sum_{x=0}^{1}\sum_{y=0}^{1} f(x,y)\, \underbrace{e^{-j2\pi\left(\frac{ux}{2}+\frac{vy}{2}\right)}}_{e^{-j\pi(ux+vy)} = e^{-j\pi y} = \begin{cases} 1; & \text{if } y=0 \\ -1; & \text{if } y=1 \end{cases}}$$

$$= f(0,0) + f(0,1)(-1) + f(1,0) + f(1,1)(-1)$$

$$= 1 + 2(-1) + 3 + 4(-1) = -2$$

Fft2

2-D fast Fourier transform

```
>> fft2(f)
ans =
  10   -2
```

$= \Big[\ F(0,0) \quad F(0,1)\ \Big]$

ans =

$$\begin{matrix} 10 & -2 \\ -4 & 0 \end{matrix} = \begin{bmatrix} F(0,0) & F(0,1) \\ F(1,0) & F(1,1) \end{bmatrix}$$

Using 1-D DFT on the rows, followed by the 1-D DFT on the columns of the DFT coefficients, obtained in the previous step.

f =

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{\text{1-D DFT}} \begin{bmatrix} [3 & -1] \\ [7 & -1] \end{bmatrix} \xrightarrow[\text{on columns}]{\text{1-D DFT}} \begin{bmatrix} 3 \\ 7 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} \xrightarrow{\text{1-D DFT}} \begin{bmatrix} 10 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \end{bmatrix} \xrightarrow{\text{1-D DFT}} \begin{bmatrix} -2 \\ 0 \end{bmatrix} \Rightarrow$$

```
>> fft2(f)
ans =
   10   -2
   -4    0
```

$= F(u,v)$

1-D DFT, where $M = 2$

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

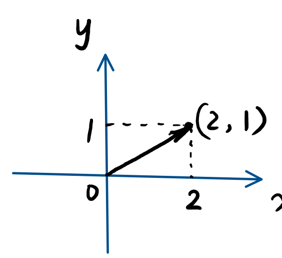
$$= \sum_{x=0}^{1} f(x) \underbrace{e^{-j\frac{2\pi ux}{2}}}_{e^{-j\pi ux} = (-1)^{ux}} = \sum_{x=0}^{1} f(x)(-1)^{ux} = f(0) + f(1)(-1)^u$$

$$\begin{cases} F(0) = f(0) + f(1)(-1)^0 = f(0) + f(1) \\ F(1) = f(0) + f(1)(-1)^1 = f(0) - f(1) \end{cases}$$

Basis Images

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) \overbrace{e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}}^{\text{Basis}}$$

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v) \underbrace{e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}}_{\text{Basis}}$$



$$(2,1) = 2\cdot\underbrace{(1,0)}_{\text{basis}} + 1\cdot\underbrace{(0,1)}_{\text{basis}}$$

$$[2 \;\; 1]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2, \quad [2 \;\; 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

For example,

If $M = N = 2$, $\quad f(x,y) = \frac{1}{4}\sum_{u=0}^{1}\sum_{v=0}^{1} F(u,v) \underbrace{e^{j\pi(ux+vy)}}_{(-1)^{ux+vy}}$ ← basis image

$$f = \begin{array}{c|cc} & \rightarrow y & \\ \downarrow x & 1 & 2 \\ & 3 & 4 \end{array}$$

$$(-1)^{ux+vy} \leftarrow \text{basis image} \quad B(x,y;u,v)$$

(1) $u=0, \quad v=0 \qquad (-1)^{ux+vy} = (-1)^0 = 1 \qquad$ Basis Image $= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B(x,y;0,0)$

(2) $u=0, \quad v=1 \qquad (-1)^{ux+vy} = (-1)^y \qquad B(x,y;0,1) = \begin{array}{c} \rightarrow y \\ \downarrow x \end{array} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(3) $u=1, \quad v=0 \qquad (-1)^{ux+vy} = (-1)^x \qquad B(x,y;1,0) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

(4) $u=1, \quad v=1 \qquad (-1)^{ux+vy} = (-1)^{x+y} \qquad B(x,y;1,1) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

```
>> fft2(f)
ans =
   10   -2
   -4    0
```

$$= F(u,v) = \begin{bmatrix} F(0,0) & F(0,1) \\ F(1,0) & F(1,1) \end{bmatrix}$$

Reconstruction:

$$10 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + (-4) \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$