Lecture 10

- Entropy Estimation (cont'd)

Conditional Prob's:

Use another example : Consider binary sequence : w w b b b b b w w

Alphabet = {b, w}

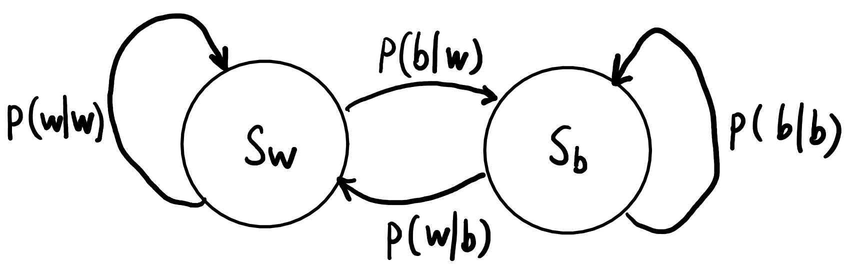
$P(w|w) = \frac{2}{3}$, $P(b|w) = \frac{1}{3}$, $P(w|b) = \frac{1}{5}$, $P(b|b) = \frac{4}{5}$

next symbol   current symbol

In comparison,

P(w,w) = 2/8 NOT Equal to P(w|w). …

- Markov Model (Chain)
  Binary source: {w,b} or {0,1}.



Suppose for another binary source:
P(Sw) = 30/31, P(Sb) = 1/31;
P(w|w) = 0.99, P(b|w) = 0.01;
P(b|b) = 0.7,   P(w|b) = 0.3

H(30/31, 1/31) = 0.2056 bit/symbol

H_M = P(Sw) x H(Sw) + P(Sb) x H(Sb)

where
H(Sw) = H(P(w|w), P(b|w)) = H(0.99, 0.01) = 0.0808
H(Sb) = H(P(w|b), P(b|b)) = H(0.3, 0.7) = 0.8813

```
>> -(30/31)*log2(30/31) - (1/31)*log2(1/31)
ans =
   0.2056
```

```
>> -0.99*log2(0.99) - 0.01*log2(0.01)
ans =
   0.0808
>> -0.7*log2(0.7) - 0.3*log2(0.3)
ans =
   0.8813
```

$H_M = P(Sw) \times H(Sw) + P(Sb) \times H(Sb)$
$= 30/31 \times 0.0808 + 1/31 \times 0.8813$
$= 0.1066$ bit/symbol

In comparison,
$H_M < H(30/31, 1/31) = 0.2056$ bit/symbol

- Jensen's Inequality

If f(x) is a convex function, then E[f(X)] >= f(E[X]), where X is a random variable, and E[X] is the expected value of X.

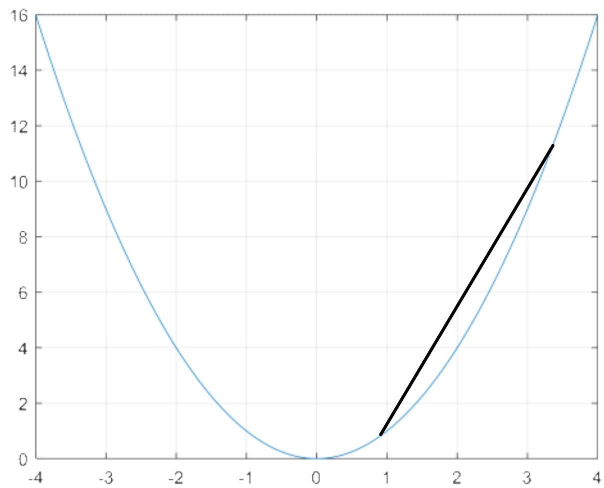
Convex: having an outline or surface curved like the exterior of a circle or sphere.

- Test if a function is convex or not.
  If f''(x) >= 0 **everywhere**, then f(x) is convex.

Example: $f(x) = x^2 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 > 0$

Thus f(x) is a strictly convex function.

$E[X^2] \geq E^2[X]$



Mean Square >= Square of the Mean

$E[X^2] - E^2[X] \geq 0$

Var[X] = E[(X - E[X])^2]

$$= E\left[X^2 - 2XE[X] + E^2[X]\right]$$

$$= E[X^2] - 2\underbrace{E[X]\cdot E[X]}_{E^2[X]} + E^2[X]$$

$$Var[X] = E[X^2] - E^2[X] \geq 0$$

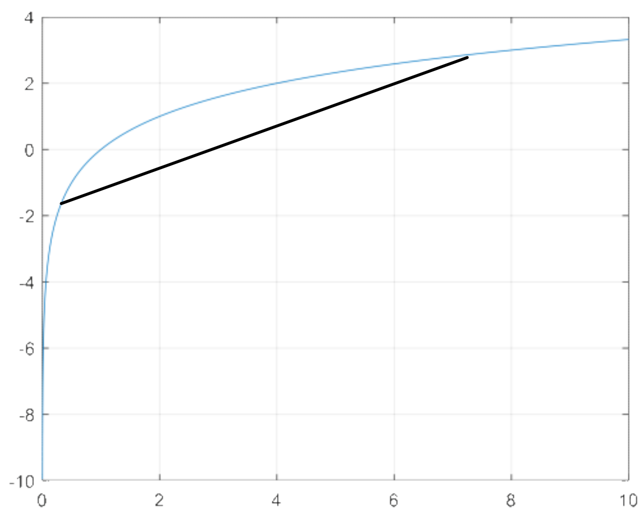
```
>> X = randn(1, 1000000);
>> mean(X.^2) - mean(X)^2
ans =
   1.0019

>> var(X)
ans =
   1.0019
```

What if the function f(x) is concave?
For example, f(x) = log$_2$(x) $= \frac{\ln x}{\ln 2}$

$$f'(x) = \frac{\frac{1}{x}}{\ln 2}, \quad f''(x) = -\frac{1}{\ln 2}\cdot\frac{1}{x^2} < 0, \quad \text{if } x \neq 0$$

$\log_2 x$ is a concave function



If f(x) is a concave function, then E[f(X)] <= f(E[X]), where X is a random variable, and E[X] is the expected value of X.

E[f(X)] <= f(E[X])

$\Downarrow$

$$E[\log_2(X)] \leq \log_2 E[X]$$

```
>> X = rand(1, 10000000);
```

```
>> mean(log2(X))
ans =
  -1.4431
```

$<$

```
>> log2(mean(X))
ans =
  -1.0002
```