Lecture 15

Kraft-McMillan Inequality (K-M inequality):

(1) If a code C is uniquely decodable, then $K(C) = \sum_{i=1}^{N} 2^{-l_i} \leqslant 1$
Where N is the number of codewords in code C, and $l_1, l_2, \ldots, l_N$
Are the codeword lengths.

(2) If $K(C) \leqslant 1$, then we can always construct a prefix code with codeword lengths being $l_1, l_2, \cdots, l_N$.

Proof of (1): Necessary condition
If a code C is uniquely decodable, then $K(C) = \sum_{i=1}^{N} 2^{-l_i} \leqslant 1$.

Otherwise, if K(C) > 1 => Proof by contradiction.

Then $[K(C)]^n$ should grow exponentially with n (n >= 2, n is an integer)

$$[K(C)]^n = \left(\sum_{i=1}^{n} 2^{-l_i}\right)^n = \left(\sum_{i_1=1}^{n} 2^{-l_{i_1}}\right)\left(\sum_{i_2=1}^{n} 2^{-l_{i_2}}\right)\cdots\left(\sum_{i_n=1}^{n} 2^{-l_{i_n}}\right)$$

$$= \sum_{k=n}^{nl} A_k \cdot 2^{-k}$$

where $l = \max\{l_1, l_2, \ldots, l_N\}$

A_k: The number of combination of n codewords that have a combined length of **k** bits.

The number of possible distinct binary sequences of length k is: 2^k

If a code C is uniquely decodable, then each sequence can represent one and only one codeword, thus $A_k \leqslant 2^k$

codeword, thus $A_K \leq 2^K$

upper bound

$$[K(C)]^n = \sum_{k=n}^{n\ell} A_k \cdot 2^{-k} \leq \sum_{k=n}^{n\ell} \underbrace{2^k \cdot 2^{-k}}_{1} = n\ell - n + 1 = n(\ell-1)+1,$$

which grows linearly with an increasing n -- a contradiction to

$[K(C)]^n$ should grow exponentially with n (n >= 2, n is an integer) if we assume K(C) > 1

In Summary, K(C) <= 1.

For example, a code C has codeword lengths :

$$\{a_1, a_2, a_3, a_4\}$$

$$\ell_1 = 1 \quad \ell_2 = 2 \quad \ell_3 = 3 \quad \ell_4 = 4$$

$n = 2$,

$$[K(C)]^2 = \left(\sum_{i=1}^{n} 2^{-\ell_i}\right)^2 = (2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}) \cdot (2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}) \quad \cdots$$

$$\begin{aligned} = \; & 2^{-(1+1)} + 2^{-(1+2)} + 2^{-(1+3)} + 2^{-(1+4)} \\ & + 2^{-(2+1)} + 2^{-(2+2)} + 2^{-(2+3)} + 2^{-(2+4)} \\ & + \cdots \\ & + 2^{-(4+4)} \end{aligned}$$

$$= 2^{-\underbrace{(1+1)}_{(1+1)}} + 2^{-\underbrace{(1+2)}_{(1+2)}} + \cdots + 2^{-\underbrace{(4+4)}_{(4+4)}}$$

length of codeword block (a_1, a_1) — length of codeword block (a_1, a_2) — length of Codeword block (a_4, a_4)

$$= \sum_{k=2}^{2\cdot\ell} A_k \cdot 2^{-k}$$

where $l = \max\{l_1, l_2, l_3, l_4\} = \max\{1, 2, 3, 4\} = 4$

A_K : The number of combination of n codewords that have a combined length of k bits.

$\{a_1, a_2, a_3, a_4\}$

$$\left.\begin{array}{ll} a_1: & 1 \\ a_2: & 01 \\ a_3: & 000 \\ a_4: & 0010 \end{array}\right\} \text{Codewords}$$

$$K(C) = 2^{-1}+2^{-2}+2^{-3}+2^{-4} < 1$$

$$= \frac{\frac{1}{2}\left(1-\frac{1}{2^4}\right)}{1-\frac{1}{2}} = 1-\frac{1}{2^4}$$

$\underbrace{(a_1, a_1)}$ block of two codewords $\rightarrow (1,1)$ 2-bit binary sequences:

$(0,0), (0,1), (1,0), (1,1)$

$$A_2 = 1 < 2^2 = 4$$

block of two codewords: 3-bit binary sequences ($2^3 = 8$ possible sequences)

$$\left.\begin{array}{l} (a_1, a_2) \longrightarrow (1,0,1) \\ (a_2, a_1) \longrightarrow (0,1,1) \end{array}\right\} \text{Only } A_3 = 2 \text{ sequences are used.}$$

Next, prove the sufficient condition.

(2) If $K(C) \leq 1$, then we can always construct a prefix code with codeword lengths being $l_1, l_2, \cdots, l_N$.

Construct a prefix code (to be cont'd)

Post Test (Midterm) Review:

Q1:
H(X): 1.8113, H(Y): 1.9056, H(X,Y): 3.3750, H(X|Y) = 1.4694, H(Y|X) = 1.5637
I(X;Y) = 0.3419, D_KL(X||Y) = 0.4906, D_KL(Y||X) = 0.5000

Q2: Code is NOT a prefix code, uniquely decodable, NOT instantaneous.

Q3: Minimum-Variance

- Break the probability tie by moving the **composite symbol upward**.
  ACL = 1.75 bits = H(X)
  Variance = 0.6875

Q4:

$P(00) = P(0|0) \cdot P(0) = 0.99 \times 0.97$

$P(01) = \ldots$

$\vdots$

$P(11) = P(1|1) \cdot P(1) = 0.7 \times 0.03$

$H(X_1, X_2) = 0.2992$

$H(X_2|X_1) = 0.1048$

$$H(X,Y) = -\sum_{x \in X}\sum_{y \in Y} p(x,y) \log_2 p(x,y)$$

$$H(X|Y) = -\sum_{x \in X}\sum_{y \in Y} p(X=x, Y=y) \log_2 p(X=x|Y=y)$$

Q5:

$$H(p, 1-p) = E[\log_2(X)] = p \cdot \log_2\left(\frac{1}{p}\right) + (1-p) \cdot \log_2\left(\frac{1}{1-p}\right)$$

$$\log_2(E[X]) = \log_2\left[p \cdot \frac{1}{p} + (1-p) \cdot \frac{1}{1-p}\right] = \log_2 2 = 1$$

$$H(p, 1-p) \lesssim 1.$$