Lecture 21

Entropy of the random variable X, with geometric distributions

RV with **Geometric** Distribution
X: takes non-negative integer values (n)

$$G(n) = p(X=n) = p^n(1-p), \quad \text{where } 0<p<1 : \text{given parameter}$$

H(X) = H{1-p, p(1-p), ..., p^n(1 - p), ... }

$$= -\sum_{n=0}^{\infty} p^n(1-p)\,\underbrace{\log_2\left[p^n(1-p)\right]}_{n\log_2 p + \log_2(1-p)}$$

$$= -\sum_{n=0}^{\infty} p^n(1-p)\, n\log_2 p - \sum_{n=0}^{\infty} p^n(1-p)\log_2(1-p)$$

$$= -(1-p)\log_2 p \cdot \sum_{n=0}^{\infty} np^n - (1-p)\log_2(1-p)\sum_{n=0}^{\infty} p^n$$

where

$$\sum_{n=1}^{\infty} np^n = \sum_{n=0}^{\infty} np^n = p + 2p^2 + \cdots + \cdots = \frac{p}{(1-p)^2}$$

(See Lecture 5)

$$\sum_{n=0}^{\infty} p^n = 1 + p + p^2 + \cdots + \cdots = \frac{1}{(1-p)}$$

e.g., $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \cdots = (1.11111\cdots1\cdots)_2$

$\frac{1}{8}$

$(0.1)_2 \quad (0.01)_2 \quad (0.001)_2$

$$S = \sum_{n=0}^{\infty} p^n$$

$$pS = \quad p + p^2 + \cdots + \cdots \qquad (1)$$

$$S = 1 + p + p^2 + \cdots + \cdots \qquad (2)$$

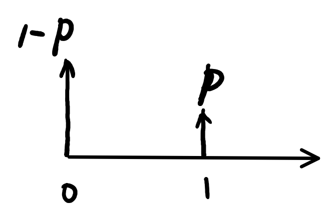
(2) - (1): $(1-p)S = 1 \Rightarrow S = \frac{1}{1-p}$

$$H(X) = -(1-p)\log_2 p \cdot \sum_{n=0}^{\infty} n p^n - (1-p)\log_2(1-p) \sum_{n=0} p^n$$

$$= -(1-p)\log_2 p \, \frac{p}{(1-p)^2} - (1-p)\log_2(1-p) \cdot \frac{1}{1-p}$$

$$= \frac{-p\log_2 p}{1-p} - \log_2(1-p)$$

$$\boxed{H(X) = \frac{-p\log_2 p - (1-p)\log_2(1-p)}{1-p}} = \frac{H(p, 1-p)}{1-p}$$



$p = \frac{1}{2}$

1. (20 pts) A fair coin is flipped until the first tail occurs. Let X denote the number of flips required. Find the entropy $H(X)$ in bits.

2 bits

$$\underbrace{HH\cdots H}_{n-1}\underset{1}{T}$$

$$P(X=n) = \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^n, \quad \text{where } n \geq 1$$

Compared with:

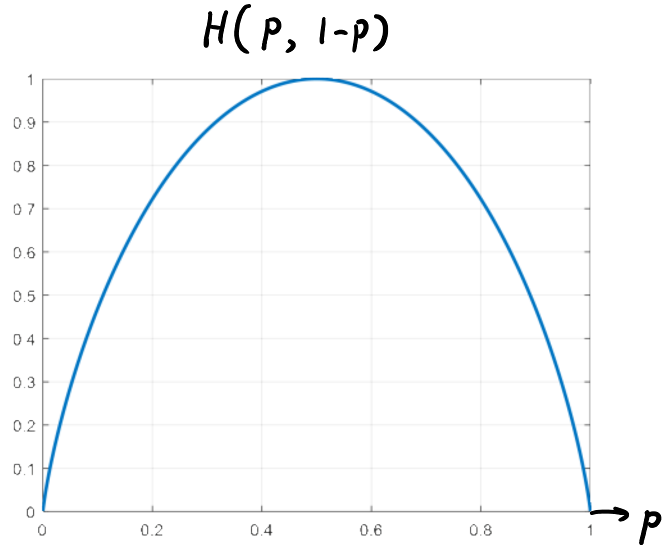
$G(n) = P(X=n) = p^n(1-p)$, where $0<p<1$: given parameter

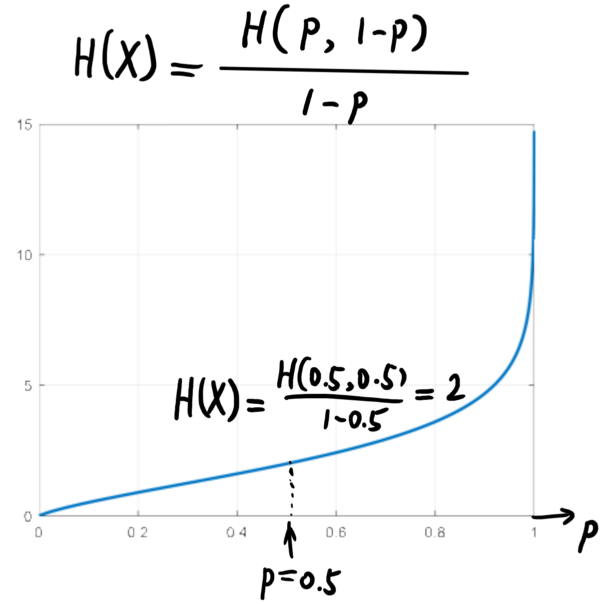
If $p = \frac{1}{2}$, then $G(n) = \left(\frac{1}{2}\right)^n\left(1-\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{n+1}$, where $n \geq 0$

$$H(X) = \frac{H(\frac{1}{2}, \frac{1}{2})}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2 \text{ bits}$$

$H(p, 1-p)$

$H(X) = \frac{H(p, 1-p)}{}$

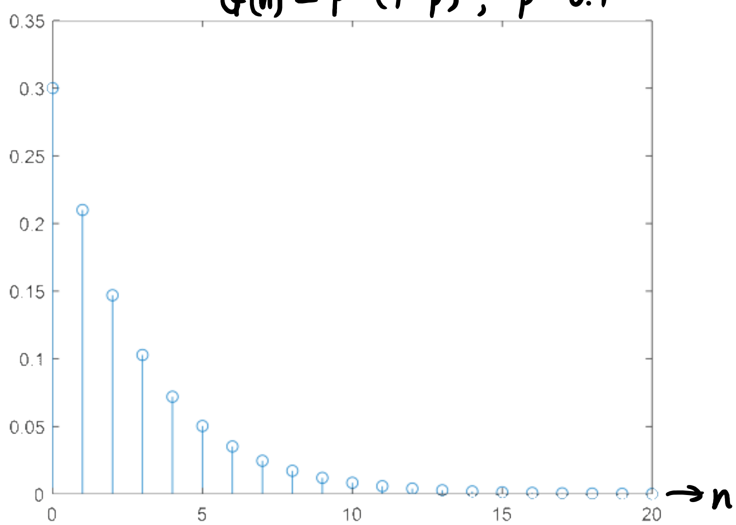

H(p, 1-p)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.2
0.4
0.6
0.8
p


H(X) = H(p, 1-p) / 1-p
H(X) = H(0.5, 0.5) / 1-0.5 = 2
p=0.5
0
5
10
15
0.2
0.4
0.6
0.8
1
p

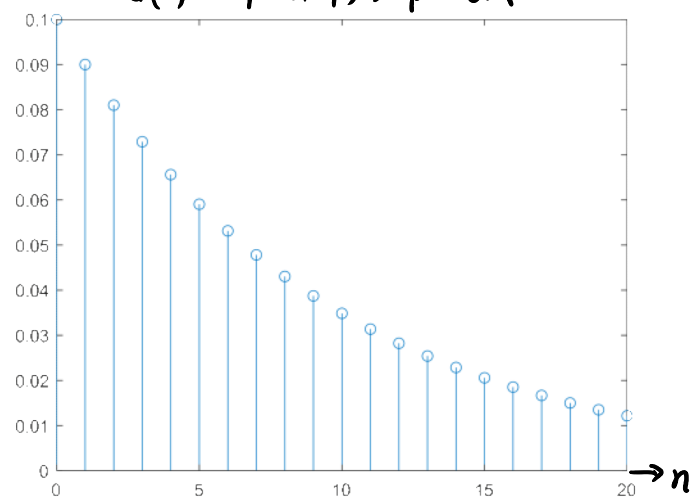
```
>> HX = H./(1-p);
>> figure;
>> plot(p,HX); grid
```

```
>> p = 0.0001: 0.0001: 0.9999;
>> H = -p.*log2(p) - (1 - p).*log2(1-p);
>> plot(p, H); grid
```

$G(n) = p^n(1-p)\,,\quad p = 0.7$



$G(n) = p^n(1-p)\,,\quad p = 0.9$




```
>> p = 0.7;
>> G = p.^n.*(1-p);
>> figure;
>> stem(n, G);
```

```
>> p = 0.9;
>> G = p.^n.*(1-p);
>> figure;
>> stem(n, G);
```

$G(n) = p^n(1-p) = \left(\frac{1}{2}\right)^{n+1}$

$m = 1 \rightarrow p^m = \frac{1}{2} \Rightarrow p = \frac{1}{2}$

n	$G(n)$	Codeword
0	1/2	0
1	1/4	10
2	1/8	110
3	1/16	1110
4	1/32	11110
5	1/64	111110
6	1/128	1111110
7	1/256	11111110
8	1/512	111111110
9	1/1024	1111111110
10	1/2048	$\underbrace{11111111110}_{10\text{ 1's}}$

In this special case, the Golomb code becomes the unary code.

Reason: there is no remainder m => NULL

$\log_2 m = \log_2 1 = 0$ bit

ACL of the unary code when $p = \frac{1}{2}$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \cdot (n+1) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m} \cdot m = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

$$= H(X)$$

For p = 1/2, Unary Code is optimal.