Lecture 25

Information Theoretic Analysis of Lossy Compression

Lossy Compression

Input Source $X \longrightarrow$ Encoder $\longrightarrow$ 3-bit words $\{0, 1, \cdots, 7\}$

4-bit words $\{0, 1, 2, \cdots, 15\}$

Shift out the LSB of the input symbol

Example

$$(4)_{10} = (0100)_2 \longrightarrow (010)_2 = (2)_{10}$$

dropped by the encoder

$$(5)_{10} = (0101)_2 \nearrow$$

Input $X = \begin{cases} 4 \\ 5 \end{cases}$ $\rightarrow$ Same Output : 2

3-bit words $\{0\ 1\ \cdots\ 7\} \longrightarrow$ Decoder $\longrightarrow Y$ Reconstructed 4-bit words

Shift in 0 as the LSB

Example

$$(010)_2 \longrightarrow (0100)_2 = (4)_{10}$$

<u>Codec</u>:

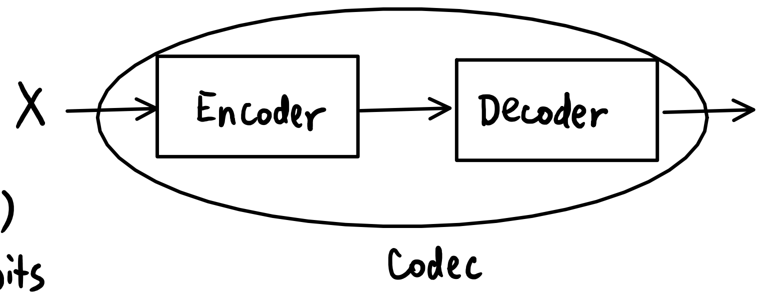
$$(5)_{10} \longrightarrow \boxed{\text{Encoder}} \xrightarrow{(010)_2} \boxed{\text{Decoder}} \longrightarrow (4)_{10}$$

Entropies:

Assume that the source X has a uniform distribution:

$$P[X=0] = P[X=1] = \cdots = P[X=15] = \frac{1}{16}$$

$$H(X) = -\sum_{i=0}^{15} P[X=i] \log_2 P[X=i] = 4 \text{ bits/symbol}$$



$H(X) = 4$ bits

Question: $H(Y) = ?$

X:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Y:	0		2		4		6		...						14	

$$P[Y=0] = P[X=0] + P[X=1] = \frac{1}{8}$$

$$P[Y=2] = P[X=2] + P[X=3] = \frac{1}{8}$$

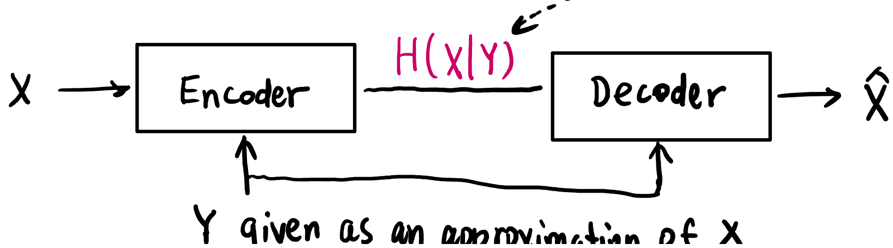
$$\vdots$$

$$P[Y=14] = \cdots = \frac{1}{8}$$

$$H(Y) = -\sum_{i=0}^{14} P[Y=i] \log_2 P[Y=i] = H\left(\frac{1}{8}, \frac{1}{8}, \cdots, \frac{1}{8}\right) = 3 \text{ bits/symbol}$$

If lossless coding is used, then H(X) bits would be needed to represent X.

If lossy coding (compression) is used, how many bits are required to represent X?



$$H(X|Y) < H(X)$$

Y given as an approximation of X

Go back to the relation between joint entropy and mutual information:

$$\boxed{H(X,Y) = H(X) + H(Y|X) = H(X) + H(Y) - I(X;Y)}$$

Where $I(X;Y) = I(Y;X) = H(X) - H(X|Y) = H(Y) - \mathbf{H(Y|X) \Rightarrow H(Y|X) = H(Y) - I(X;Y)}$

Ans: $H(X|Y) = ?$

$$H(X|Y) = -\sum_{x\in X}\sum_{y\in Y} p(X=x, Y=y)\log_2 p(X=x|Y=y)$$

where

$$p(X=i|Y=j), \text{ where } \quad i = \underbrace{0,1,\cdots,15}_{16 \text{ symbols}}, \quad j = \underbrace{0,2,\cdots,14}_{8 \text{ symbols}}$$

$$= \begin{cases} \frac{1}{2} & , \text{ if } i=j, \text{ or } i=j+1 \\ 0 & , \text{ otherwise} \end{cases}$$

X: 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Y: 0 2 4 6 ... 14

e.g., $\begin{cases} p(X=0|Y=0) = \frac{1}{2} \\ p(X=1|Y=0) = \frac{1}{2} \end{cases}$

Next,

$$p(X=x, Y=y) = p(X=i|Y=j)\cdot \underbrace{p(Y=j)}_{\frac{1}{8}}, \text{ By Bayes Rule}$$

Thus

$$H(X|Y) = -\sum\sum p(X=i|Y=j)\cdot \underbrace{p(Y=j)}_{\frac{1}{8}}\cdot \log_2 p(X=i|Y=j)$$

$$H(X|Y) = -\sum_{j=0}^{14} \underbrace{p(X=j|Y=j)}_{\frac{1}{2}} \cdot \underbrace{p(Y=j)}_{\frac{1}{8}} \cdot \log_2 \underbrace{p(X=j|Y=j)}_{\frac{1}{2}}$$

$$- \sum_{j=0}^{14} \underbrace{p(X=j+1|Y=j)}_{\frac{1}{2}} \cdot \underbrace{p(Y=j)}_{\frac{1}{8}} \cdot \log_2 \underbrace{p(X=j+1|Y=j)}_{\frac{1}{2}}$$

$$= -8\left(\frac{1}{16} \times \log_2 \frac{1}{2}\right) - 8\left(\frac{1}{16} \times \log_2 \frac{1}{2}\right)$$

$$= 1 \text{ bit}$$

I(Y;X) = H(X) - H(X|Y) $= 4 - 1 = 3$ bits

I(X;Y) = I(Y;X) = H(X) - H(X|Y) = H(Y) - **H(Y|X) => H(Y|X) = H(Y) - I(X;Y)**

$H(Y|X) = H(Y) - I(X;Y) = 3 - 3 = 0$ => Given X, there is NO uncertainty about Y.