Lecture 5

Example: RV with **Geometric** Distribution

X: takes non-negative integer values (n)

$$G(n) = P(X=n) = p^n (1-p), \quad \text{where } 0<p<1 : \text{given parameter}$$

Assume that the probability of a symbol '0' occurring is: $p \Rightarrow P('1') = 1-p$ for binary source

$$P(\underbrace{0\,0 \cdots 0}_{\text{run length} = n}\,1) = p^n \cdot (1-p)$$

geornd

Geometric random numbers

The geometric distribution is useful to model the number of failures before one success in a series of independent trials, where each trial results in either success or failure, and the $\underbrace{\textbf{probability of success}}_{(1-p)}$ in any individual trial is the constant p.

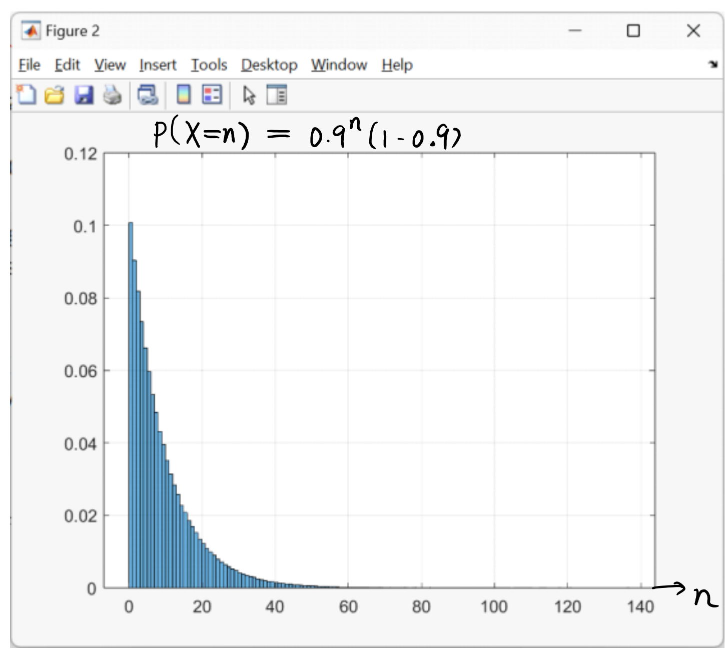
```
>> X = geornd(1 - 0.9, 1, 1000000);
>> whos X
  Name      Size              Bytes  Class     Attributes

  X       1x1000000          8000000  double

>> doc histogram
>> figure; histogram(X,138,'Normalization','pdf');
>> grid

>> Y = find(X == 1);
>> length(Y)/length(X)
ans =
    0.0897
```

$$p(X=1) = 0.9^1 \times (1-0.9) = 0.9 \times 0.1 = 0.09$$

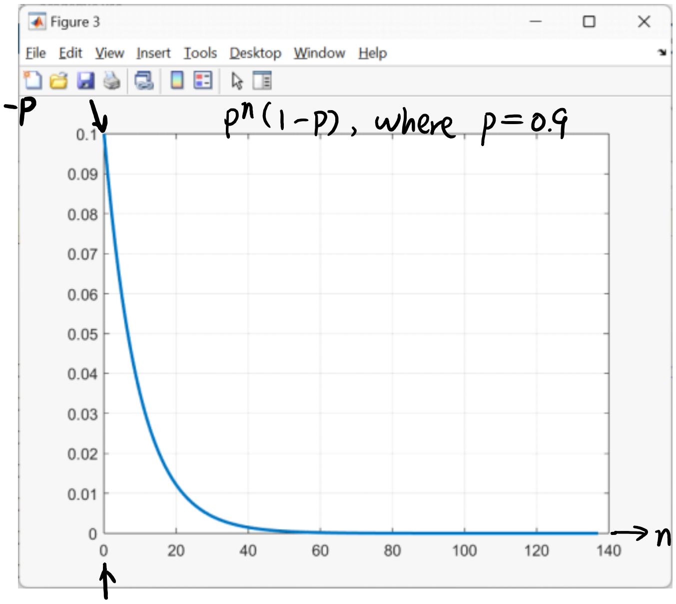



```
>> max(X)
ans =
   137
```

```
>> min(X)
ans =
     0
```

```
>> p = 0.9;
>> n = 0: 137;
>> G = p.^n*(1 - p);
>> figure; plot(n, G); grid
```

$1-p$



$n=0$

$p^n(1-p)=1-p$

Next, determine the mean E[X]:

$$G(n) = p^n(1-p) = P[X=n]$$

$$E[X] = \sum_{n=0}^{\infty} n \cdot P[X=n],$$

$$= \sum_{n=0}^{\infty} n \cdot p^n(1-p) = (1-p)\underbrace{\sum_{n=0}^{\infty} np^n}_{p+2p^2+3p^3+4p^4+\cdots}$$

$$S = p+2p^2+3p^3+4p^4+\cdots \qquad (1)$$

$$pS = \qquad p^2+2p^3+3p^4+\cdots \qquad (2)$$

(1) − (2):

$$(1-p)S = p+p^2+p^3+p^4+\cdots = \frac{p}{1-p}, \qquad 0<p<1$$

$$S = \frac{p}{(1-p)^2}$$

Thus $E[X] = \dfrac{p}{1-p} = \dfrac{1}{\frac{1}{p}-1}$, $p\downarrow \Rightarrow \frac{1}{p}\uparrow \Rightarrow E[X]\downarrow$

```
>> mean(X)

ans =

    8.9859
```

If $p=0.9$, $E[X] = \dfrac{0.9}{1-0.9} = 9$

$p=0.1$, $E[X] = \dfrac{0.1}{1-0.1} = \dfrac{0.1}{0.9} = 0.11..$

Information Theory:

(1) Self Information (of an event A, which occurs with a probability P(A))

$$i(A) = \log_2 \frac{1}{P(A)} = -\log_2 P(A)$$

$$P(A)\uparrow \Rightarrow i(A)\downarrow$$

Self Information of two events A and B

$$i(AB) = \log_2 \frac{1}{P(AB)}$$

If A and B are independent, then P(AB) = P(A) x P(B); Also, P(A) = P(A|B), or P(B) = P(B|A)

If A and B are **not** independent, then P(AB) = P(A) x P(B|A) = P(B) x P(A|B): Bayes Rule

If A and B are independent, $i(AB) = \log_2\left[\frac{1}{P(A)}\cdot\frac{1}{P(B)}\right] = \log_2\frac{1}{P(A)} + \log_2\frac{1}{P(B)}$

$$= i(A) + i(B)$$

Example: Flip a biased coin, P(H) = 1/8, P(T) = 7/8.

i(H) = $\log_2 \frac{1}{1/8} = 3$ bits

i(T) = $\log_2 \frac{1}{7/8} = 0.1926$ bit

```
>> log2(8/7)
ans =
   0.1926
```

- Source Entropy: Everage Self Information (uncertainty about the source X)

$$\underbrace{\text{Source}}_{X}$$

$$H(X) = \sum_i p(A_i)\cdot i(A_i) = -\sum_i p(A_i)\cdot\log_2 p(A_i)$$

```
>> 3/8 + 7/8*0.1926
ans =
```

Example: Flip a biased coin, P(H) = 1/8, P(T) = 7/8.

$$H(X) = P(H)\cdot i(H) + P(T)\cdot i(T) = \frac{1}{8}\times 3 + \frac{7}{8}\times 0.1926 = 0.5435 \text{ bit}$$