Lecture 7

HW2 posted:
Bit-plane sequences and analysis of run-length coding on binary bit-plane sequences.

Source Entropy for two random variables X and Y. (cont'd)
Joint Entropy: Uncertainty about the joint sources X and Y.

$$H(X,Y) = -\sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(x,y)$$

$$H(X|Y) = -\sum_{x\in X}\sum_{y\in Y} p(X=x, Y=y)\log_2 p(X=x|Y=y) = \frac{11}{8} \text{ bits} < H(X)$$

$$H(Y|X) = -\sum_{x\in X}\sum_{y\in Y} p(X=x, Y=y)\log_2 p(Y=y|X=x) = \frac{13}{8} \text{ bits} < H(Y)$$

$$H(X) = \frac{7}{4} \text{ bits}$$

$$H(Y) = 2 \text{ bits}$$

Joint Distribution P(X,Y):

Y \ X	1	2	3	4	Σ
1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$	$P(Y=1) = \frac{1}{4}$
2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{32}$	$P(Y=2) = \frac{1}{4}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$P(Y=3) = \frac{1}{4}$
4	$\frac{1}{4}$	0	0	0	$P(Y=4) = \frac{1}{4}$
Σ	$P(X=1) \Downarrow \frac{1}{2}$	$P(X=2) \Downarrow \frac{1}{4}$	$P(X=3) \Downarrow \frac{1}{8}$	$P(X=4) \Downarrow \frac{1}{8}$	

- Mutual Information I(X;Y)

$$I(X;Y) = \sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2\left[\frac{p(x|y)}{p(x)}\right] \triangleq \sum_{x\in X}\sum_{y\in Y} p(X=x,Y=y)\log_2\left[\frac{p(X=x|Y=y)}{P(X=x)}\right]$$

$$= \sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(x|y) - \underbrace{\sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(x)}_{\Downarrow\ \sum_{x\in X} p(x)\log_2 p(x),\ \text{where } \sum_{y\in Y} p(x,y) = p(x)}$$

$$= -H(X|Y) + H(X)$$

In summary, I(X;Y) = H(X) - H(X|Y), or
H(X) = I(X;Y) + H(X|Y)

On the other hand,

- Mutual Information I(X;Y)

$$I(X;Y) = \sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2\left[\frac{p(x|y)}{p(x)}\right] = \sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2\left[\frac{p(x,y)}{p(x)\,p(y)}\right]$$

$$\text{Bayes' Rule} \Rightarrow \frac{p(x|y)\,p(y)}{p(x)\,p(y)} = \frac{p(x,y)}{p(x)\cdot p(y)} = \frac{p(y|x)\cdot p(x)}{p(x)\cdot p(y)} = \frac{p(y|x)}{p(y)}$$

- Mutual Information I(X;Y) can be defined also as:

$$I(Y;X) = \sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2\left[\frac{p(y|x)}{p(y)}\right]$$

$$= \sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(y|x) - \sum_{x\in X}\sum_{y\in Y} p(x,y)\log_2 p(y)$$

$$= -H(Y|X) + H(Y)$$

In summary, I(X;Y) = H(Y) - H(Y|X)
Previously, I(X;Y) = H(X) - H(X|Y)

Overall,

$$\boxed{I(X;Y) = I(Y;X) = H(X) - H(X|Y) = H(Y) - H(Y|X)}$$

$H(X) = \frac{7}{4}$ bits

$I(X;Y) = \frac{3}{8}$ bits

$H(X|Y) = \frac{11}{8}$ bits

$H(Y) = 2$ bits

$I(Y;X) = \frac{3}{8}$ bits

$H(Y|X) = \frac{13}{8}$ bits

$H(X|Y) = \frac{11}{8}$ bits

$H(Y|X) = \frac{13}{8}$ bits