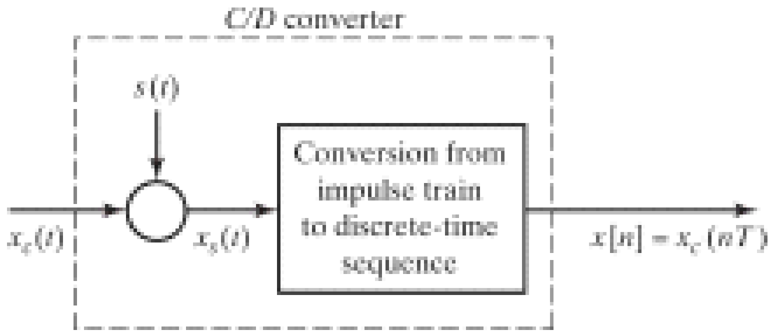
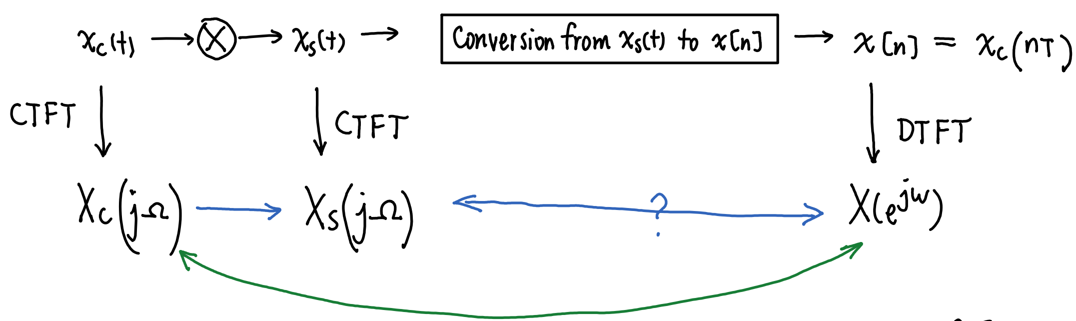
Lecture 10





$$x_s(t) = x_c(t) \cdot s(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} \overbrace{x_c(nT)}^{x[n]} \delta(t-nT)$$

CTFT of $x_c(t)$: $X_c(j\Omega)$

CTFT of $x_s(t)$: $X_s(j\Omega) = \mathcal{F}T\left\{ \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT) \right\}$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, \underbrace{\mathcal{F}T\{\delta(t-nT)\}}$$

$$\delta(t-nT) \xrightarrow{CTFT} \int_{-\infty}^{\infty} \delta(t-nT)\, e^{-j\Omega t}\, dt = e^{-j\Omega nT}$$

Thus

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} \overbrace{x_c(nT)}^{x[n]} e^{-j\Omega nT}$$

How about the DTFT of $x[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega = \Omega T}$$

$$= X(e^{j\Omega T})$$

Relation : $\boxed{\Omega T \rightarrow \omega \quad \text{or} \quad \frac{\omega}{T} \rightarrow \Omega}$

Ω : rad/sec

ω : rad

In Lecture 9,

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left[j(\Omega - k \cdot \Omega_s)\right] \quad , \quad \text{where } \Omega_s = \frac{2\pi}{T}$$

$$\Omega \leftarrow \frac{\omega}{T} \qquad \Omega_s \leftarrow \frac{2\pi}{T}$$

$$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega = \Omega T}$$

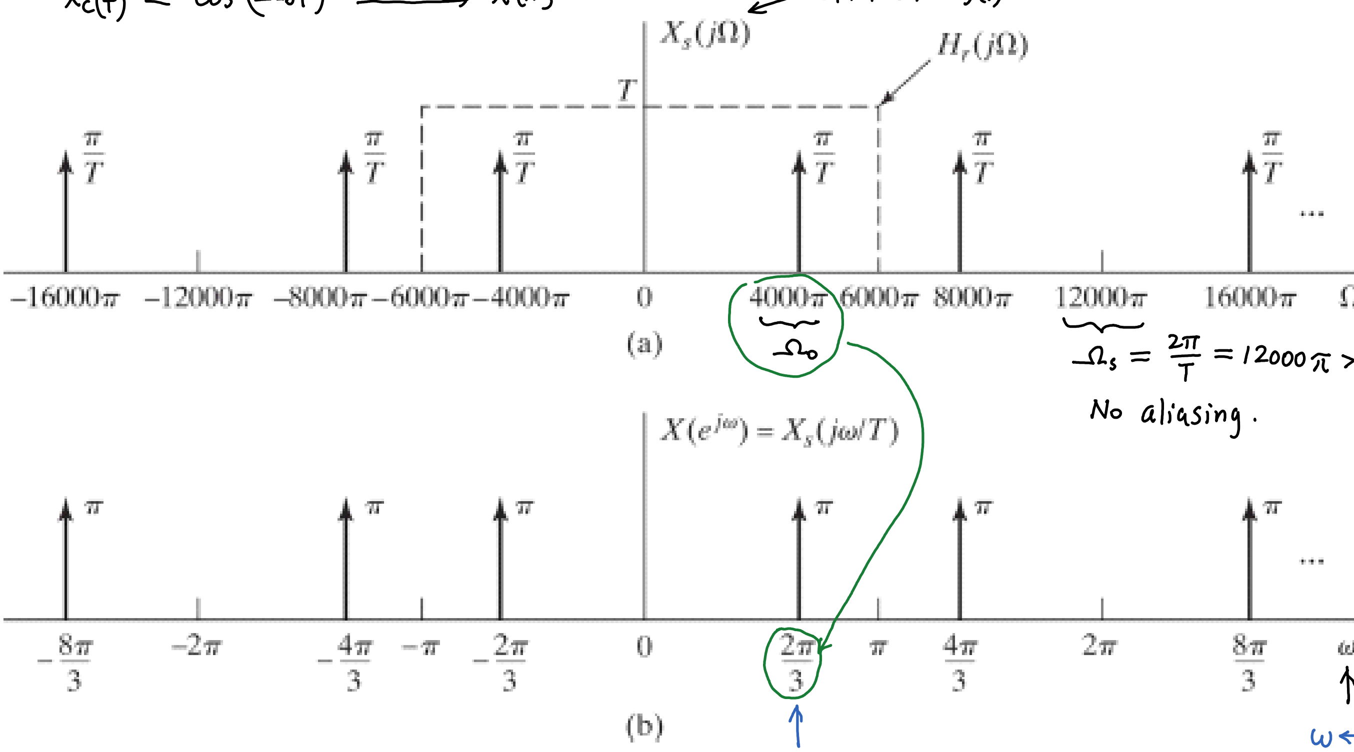
$$\underbrace{X\left(e^{j\underbrace{\Omega T}_{\omega}}\right)}_{\text{DTFT of } x[n]} = X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right)\right]$$

$$\downarrow$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

Figure 4.6 (a) Continuous-time and (b) discrete-time Fourier transforms for sampled cosine signal with frequency $\Omega_0 = 4000\pi$ and sampling period $T = 1/6000$.

$$x_c(t) = \cos(\Omega_0 t) \xrightarrow{\text{Sampling}} x[n]$$

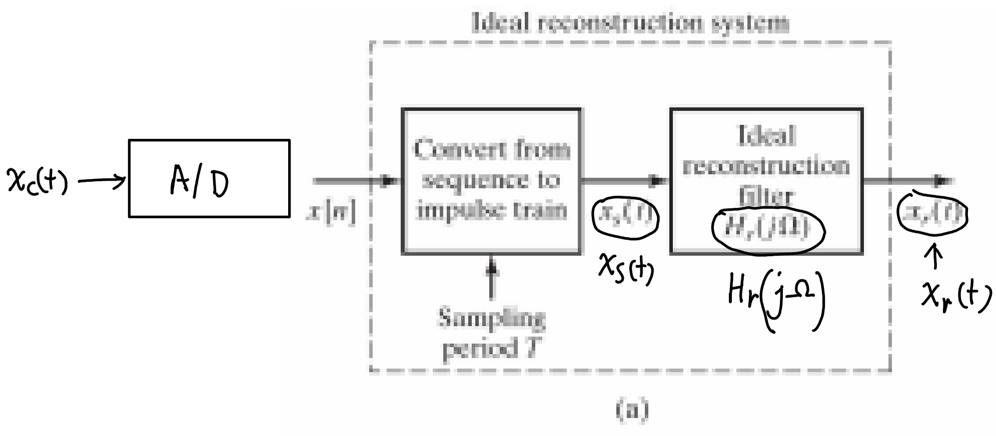


$\Omega_s = \frac{2\pi}{T} = 12000\pi > 2\Omega_0$

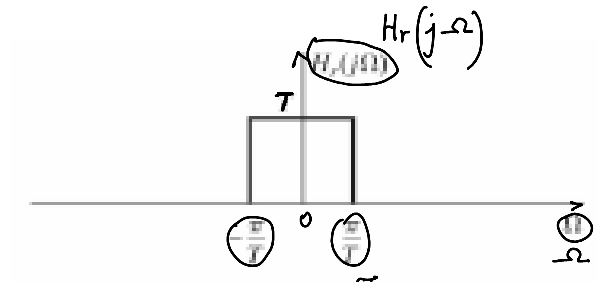
No aliasing.

$$\Omega_0 \times T = 4000\pi \times \frac{1}{6000} = \frac{2\pi}{3}$$

- Reconstruction of a Bandlimited Signal



(a)



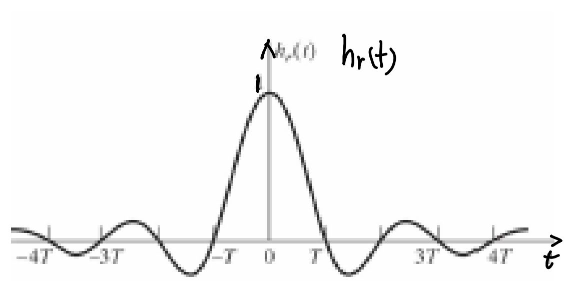
(b)

$$H_r(j\Omega) \xrightarrow{\mathcal{F}^{-1}} h_r(t)$$

Impulse Response of the reconstruction filter

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} T \cdot e^{j\Omega t}\, d\Omega = \frac{\sin\left(\frac{\pi}{T}t\right)}{\left(\frac{\pi}{T}t\right)}$$



(c)

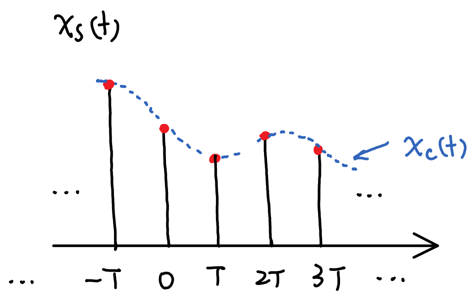
In time domain,

$$x_s(t) \rightarrow \boxed{h_r(t)} \rightarrow x_r(t)$$

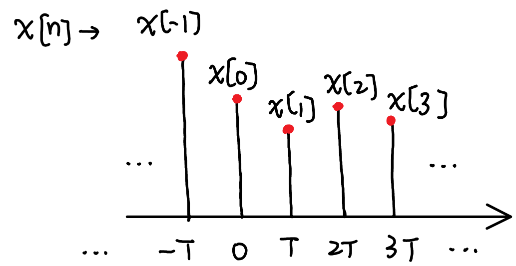
ideal LPF

$$x_r(t) = x_s(t) * h_r(t) = \int_{-\infty}^{\infty} x_s(\tau) \cdot h_r(t-\tau)\, d\tau$$

where $x_s(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \delta[t-nT]$



$\Leftarrow$



$$x_s(t) \rightarrow \boxed{h_r(t)} \rightarrow x_r(t)$$

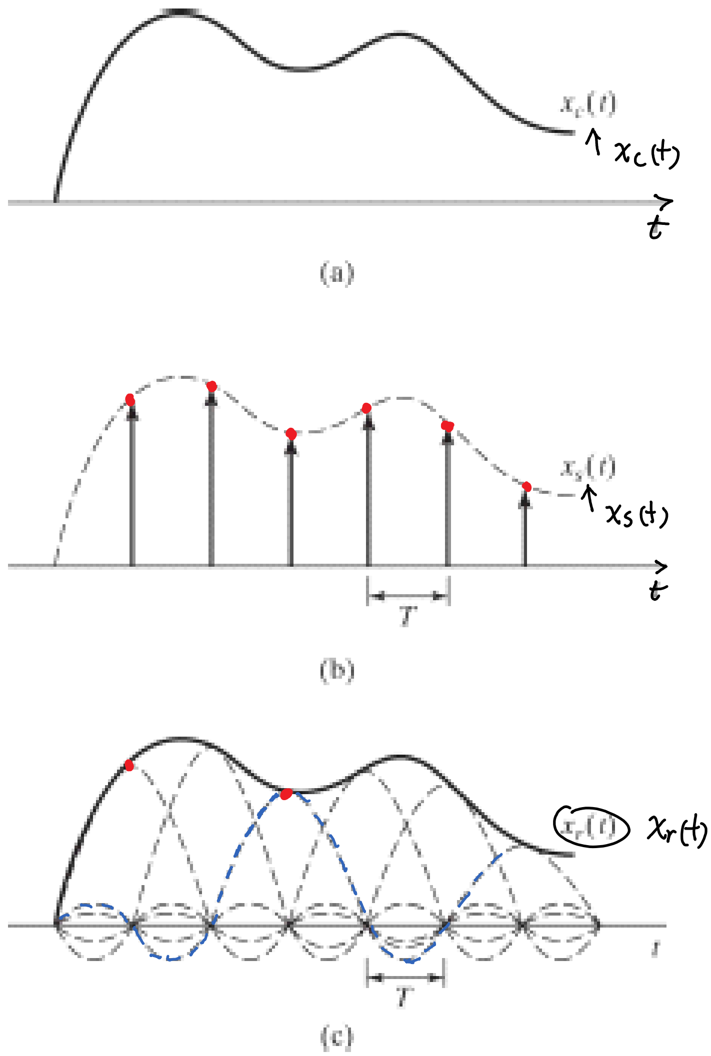
ideal LPF

$$\delta(t) \rightarrow \boxed{h_r(t)} \rightarrow h_r(t), \qquad \delta(t-T) \rightarrow \boxed{h_r(t)} \rightarrow h_r(t-T), \ \ldots$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \delta[t-nT]$$

$$\Big\downarrow \text{Ideal LPF}$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot h_r(t-nT) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\left[\frac{\pi(t-nT)}{T}\right]}{\frac{\pi(t-nT)}{T}}$$


$x_c(t)$
$x_c(t)$
t
(a)
$x_s(t)$
$x_s(t)$
t
T
(b)
$x_r(t)$
$x_r(t)$
t
T
(c)

- Frequency Domain Representation of the Ideal D/C Converter

$$x[n] \longrightarrow \boxed{D/C} \xrightarrow{x_s(t)} \boxed{\text{Filter}} \longrightarrow x_r(t)$$

$$x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega}) \qquad x_r(t) \xrightarrow{\text{CTFT}} X_r(j\Omega)$$

$$X(e^{j\omega}) \qquad x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot h_r(t-nT) \qquad X_r(j\Omega) = ?$$

$$X_r(j\Omega) \triangleq \mathcal{FT}\left\{ x_r(t) \right\} = \mathcal{FT}\left\{ \sum_{n=-\infty}^{\infty} x[n] \cdot h_r(t-nT) \right\}$$

$$= x[n] \sum_{n=-\infty}^{\infty} \mathcal{FT}\left\{ h_r(t-nT) \right\}$$

$$h_r(t) \xrightarrow{\text{CTFT}} H_r(j\Omega)$$

$$h_r(t-nT) \xrightarrow{\text{CTFT}} e^{-j\Omega nT} H_r(j\Omega)$$: Time-Shifting property of CTFT

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot H_r(j\Omega)\, e^{-j\Omega nT}$$

$$= H_r(j\Omega) \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega nT}$$

$$x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$\downarrow \quad \omega \leftarrow \Omega T$$

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega T n}$$

Hence $$X_r(j\Omega) = H_r(j\Omega) \cdot X(e^{j\Omega T}).$$