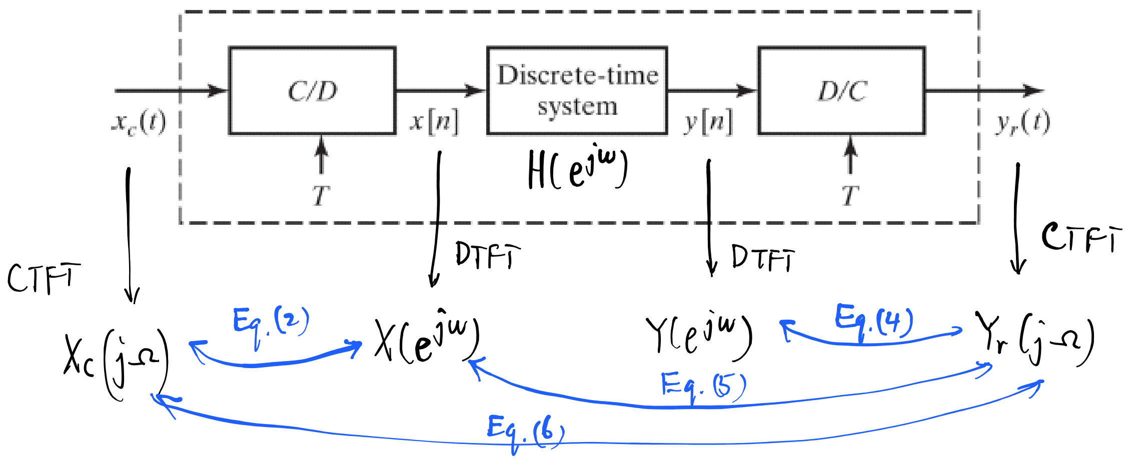
Lecture 11

Discrete-Time Processing of Continuous-Time Signals



– C/D Converter

$$x[n] = x_c(t)\Big|_{t=nT} = x_c(nT) \quad \cdots\cdots (1)$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) \quad \cdots\cdots (2)$$

– D/C Converter

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \cdot \frac{\sin\left[\frac{\pi(t-nT)}{T}\right]}{\frac{\pi(t-nT)}{T}} \quad \cdots\cdots (3)$$

$$Y_r(j\Omega) = H_r(j\Omega) \cdot Y(e^{j\Omega T}) \quad \cdots\cdots (4)$$

– D-T System has a system function $H(e^{j\omega})$

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

From (4) : $$Y_r(j\Omega) = H_r(j\Omega) \cdot H(e^{j\Omega T}) \cdot X(e^{j\Omega T}) \quad \cdots\cdots (5)$$

From (2) :

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) \quad \cdots\cdots (2)$$

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\Omega \cancel{T}}{\cancel{T}} - \frac{2\pi k}{T}\right)\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

From (5) :

$$Y_r(j\Omega) = H_r(j\Omega)\cdot H(e^{j\Omega T})\cdot X(e^{j\Omega T}) \quad \cdots\cdots (5)$$

$k=0$

$X_c(j\Omega)$: original spectrum

$$Y_r(j\Omega) = H_r(j\Omega)\cdot H(e^{j\Omega T})\cdot \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right) \quad \cdots\cdots (6)$$

Since

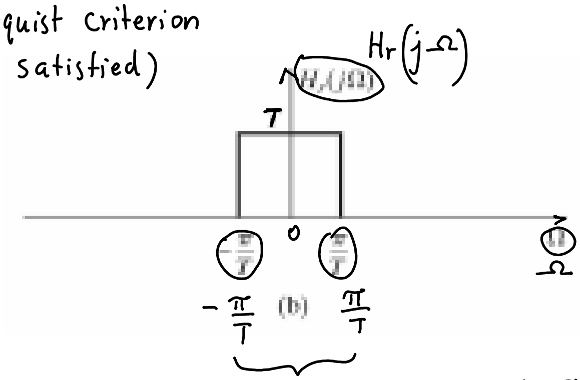
$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \frac{2\pi k}{T}\right)\right) \quad \cdots\cdots (2)$$

Assumption : The input $x_c(t)$ is bandlimited

$$X_c(j\Omega) = 0 \quad \text{if} \quad |\Omega| \geq \frac{\pi}{T}$$ (Nyquist criterion is satisfied)

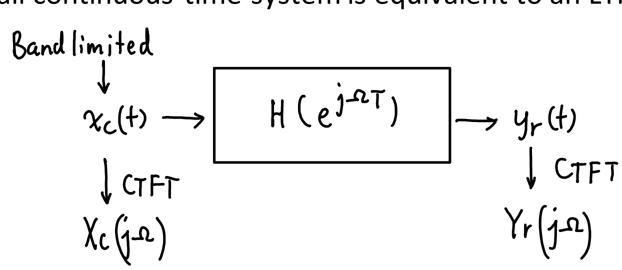
Then from (6)

$$\boxed{Y_r(j\Omega) = \begin{cases} 0, & |\Omega| \geq \frac{\pi}{T} \\ H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \frac{\pi}{T} \end{cases}}$$

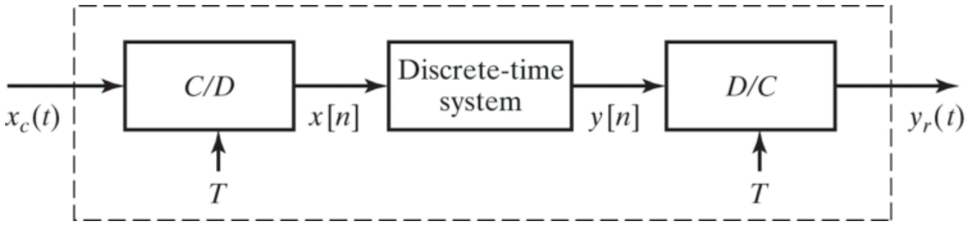


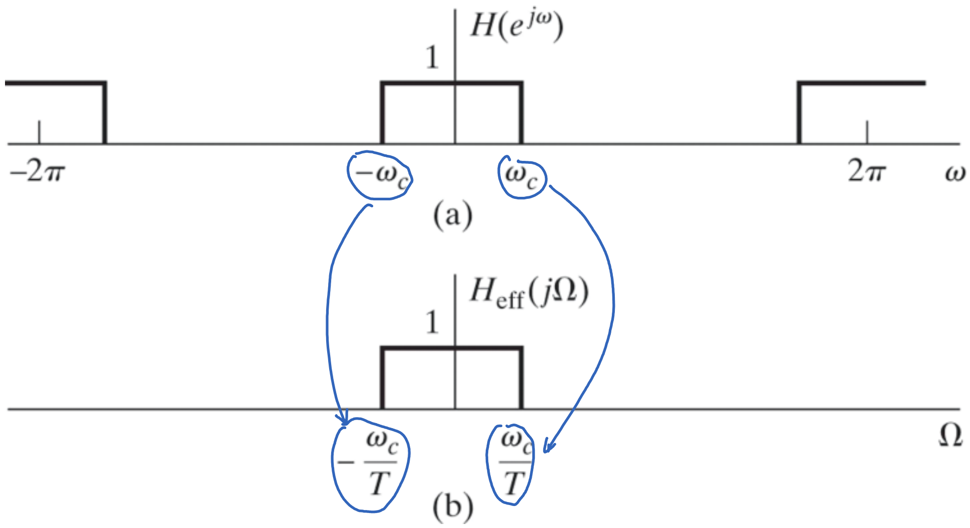
$$H_r(j\Omega) = \begin{cases} T, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| \geq \frac{\pi}{T} \end{cases}$$

The overall continuous-time system is equivalent to an LTI system:

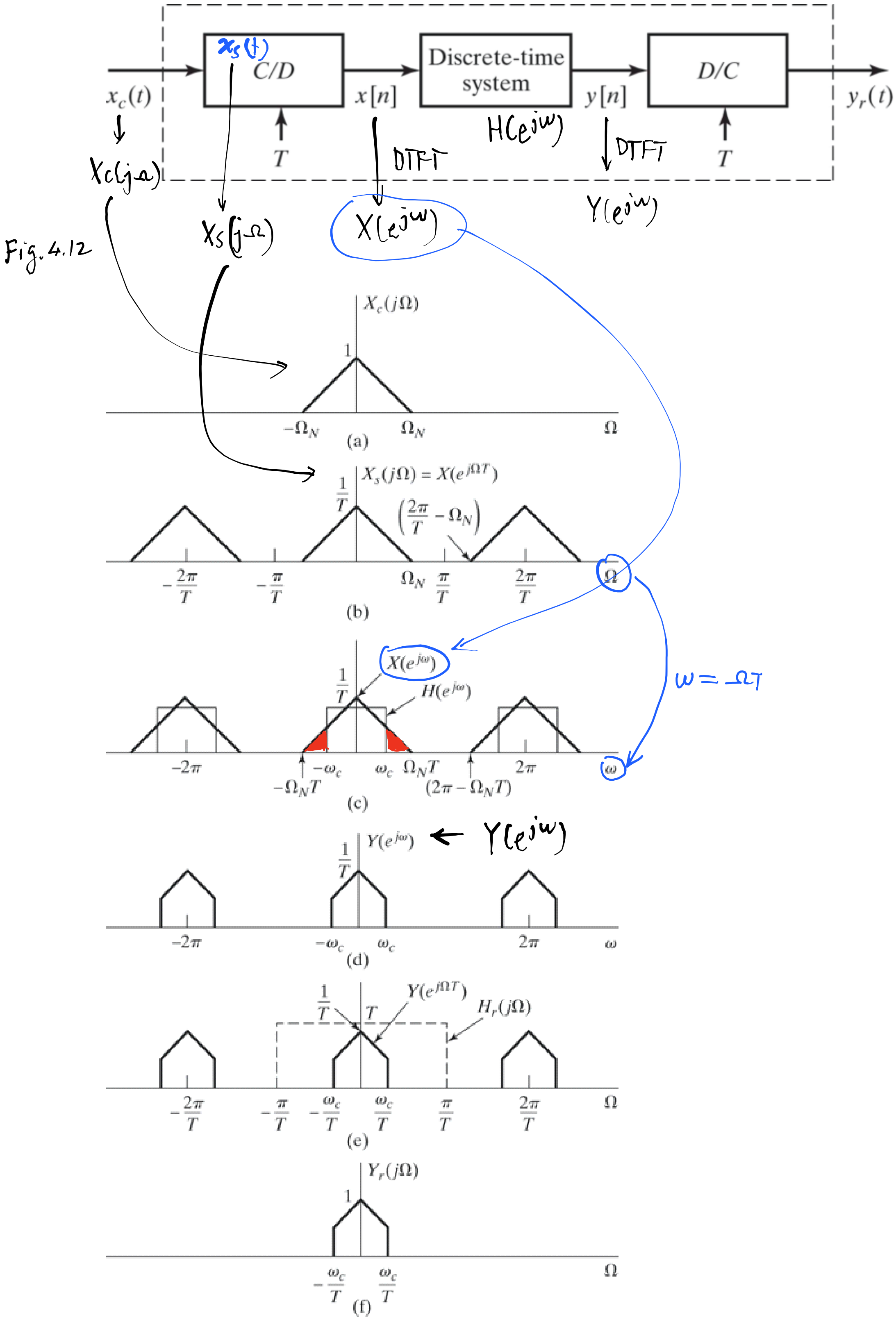


$H_{eff}(j\Omega)$: CTFT of the entire system



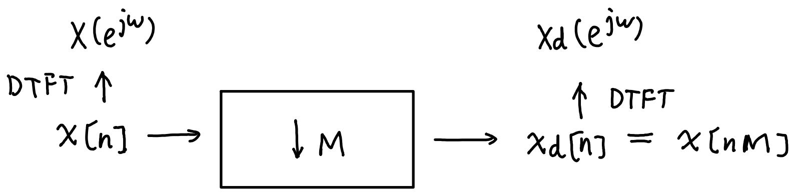


$$\boxed{Y_r(j\Omega) = \begin{cases} 0, & |\Omega| \geq \frac{\pi}{T} \\ H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \frac{\pi}{T} \end{cases}}$$

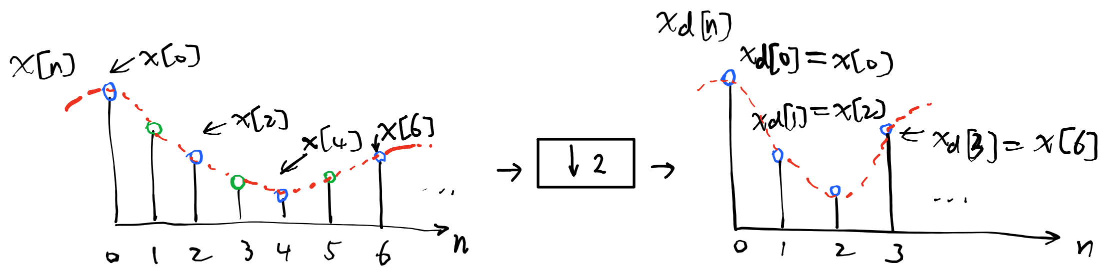


$x_s(t)$
C/D
Discrete-time system
D/C
$x_c(t)$
$x[n]$
$y[n]$
$y_r(t)$
T
T
$H(e^{j\omega})$
$X_c(j\Omega)$
DTFT
DTFT
$X_s(j\Omega)$
$X(e^{j\omega})$
$Y(e^{j\omega})$
Fig. 4.12
$X_c(j\Omega)$
1
$-\Omega_N$
Ω_N
Ω
(a)
$X_s(j\Omega) = X(e^{j\Omega T})$
$\frac{1}{T}$
$\left(\frac{2\pi}{T} - \Omega_N\right)$
$-\frac{2\pi}{T}$
$-\frac{\pi}{T}$
Ω_N
$\frac{\pi}{T}$
$\frac{2\pi}{T}$
Ω
(b)
$\omega = \Omega T$
$X(e^{j\omega})$
$H(e^{j\omega})$
$\frac{1}{T}$
-2π
$-\omega_c$
ω_c
$\Omega_N T$
2π
ω
$-\Omega_N T$
$(2\pi - \Omega_N T)$
(c)
$Y(e^{j\omega})$
$Y(e^{j\omega})$
$\frac{1}{T}$
-2π
$-\omega_c$
ω_c
2π
ω
(d)
$\frac{1}{T}$
T
$Y(e^{j\Omega T})$
$H_r(j\Omega)$
$-\frac{2\pi}{T}$
$-\frac{\pi}{T}$
$-\frac{\omega_c}{T}$
$\frac{\omega_c}{T}$
$\frac{\pi}{T}$
$\frac{2\pi}{T}$
Ω
(e)
$Y_r(j\Omega)$
1
$-\frac{\omega_c}{T}$
$\frac{\omega_c}{T}$
Ω
(f)

- Change the Sampling rate using Discrete-Time Processing

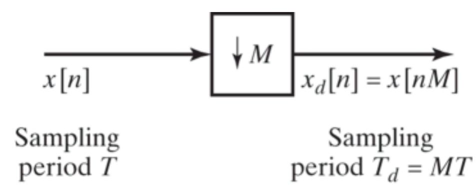


For example, $M = 2$



$$x_d[n] = x[2n]$$

In frequency domain, suppose that there exists $x_c(t)$



$$x[n] = x_c(nT)$$

$\downarrow$ DTFT

$$X(e^{jw}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right) \quad \cdots\cdots \text{Eq. (2)}$$

$$x_d[n] = x_c(nMT)$$

$\downarrow$ DTFT

?

$$X_d(e^{jw}) = \frac{1}{MT}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{w}{MT} - \frac{2\pi k}{MT}\right)\right)$$

Change the variable $k \rightarrow r$. rewrite $X_d(e^{jw})$

$$X_d(e^{jw}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{w}{MT} - \frac{2\pi r}{MT}\right)\right)$$

Let $r = i + kM$, partitioning of r such that $\begin{cases} k: \text{quotient} \\ i: \text{remainder} \end{cases}$ of $\frac{r}{M}$.

For example, if $M=2$, then $r = i + 2\cdot k$ $\rightarrow$ $\frac{r}{2} \rightarrow \begin{cases} k: \text{quotient} \\ i: \text{remainder} \end{cases}$

	r	i	k
even →	0	0	0
odd →	1	1	0
	2	0	1
	3	1	1
	4	0	2
	5	1	2
	⋮	⋮	⋮

Go back to $X_d(e^{j\omega})$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right)$$

$i + kM$

$r = i + kM$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi k \cancel{M}}{\cancel{M}T} - \frac{2\pi i}{MT}\right)\right)\right]$$

How about

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) \quad \text{...... Eq.(2)}$$

$$X(e^{j\underbrace{\frac{\omega - 2\pi i}{M}}}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{T}\right)\right)$$

Thus $\frac{\omega}{M} - \frac{2\pi i}{M}$

$$\boxed{X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right)}$$

$\frac{\omega}{M}$: scaling

$-\frac{2\pi i}{M}$: shifting

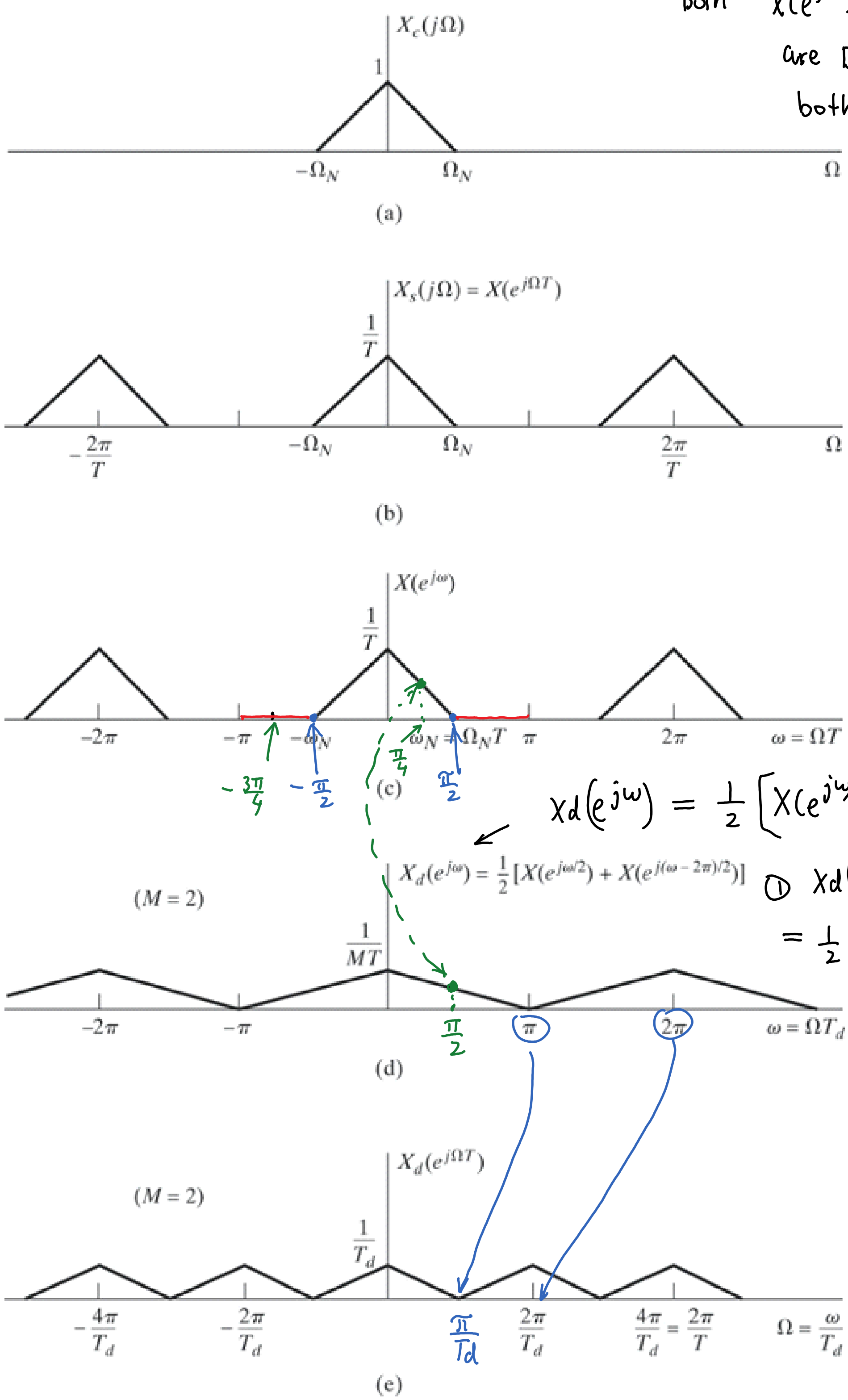
$X_d(e^{j\omega})$: M copies of the DTFT of $x[n]$, $X(e^{j\omega})$ with the frequency $\underbrace{\text{scaled}}_{\text{expansion}}$ by M and shifted by integer multiples of 2π.

$$x_c(t) \xrightarrow{\text{Sampling}} x_s(t) \xrightarrow{\text{Discretization}} x[n] \longrightarrow \boxed{\downarrow M=2} \longrightarrow x_d[n] \xrightarrow{\text{Recon. } (T_d)} x_d(t)$$

$$\downarrow \text{CTFT} \qquad \downarrow \text{CTFT} \qquad \downarrow \text{DTFT} \qquad \downarrow \text{DTFT} \qquad \downarrow \text{CTFT}$$

$$X_c(j\Omega) \qquad X_s(j\Omega) \qquad X(e^{j\omega}) \xrightarrow{\;?\;} X_d(e^{j\omega}) \qquad X_d(j\Omega)$$

Both $X(e^{j\omega})$ and $X_d(e^{j\omega})$ are DTFT's, so they are both periodic with fundamental period $= 2\pi$.



$$X_d(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega/2}) + X(e^{j(\frac{\omega}{2}-\pi)})\right]$$

① $X_d(e^{j\pi})$ if $\omega = \pi$

$$= \frac{1}{2}\left[X(e^{j\frac{\pi}{2}}) + X(e^{j(\frac{\pi}{2}-\pi)})\right]$$

$$= \frac{1}{2}[0 + 0] = 0$$

② $X_d(e^{j\frac{\pi}{2}})$ if $\omega = \frac{\pi}{2}$

$$= \frac{1}{2}\left[X(e^{j\frac{\pi}{4}}) + X(e^{j(\frac{\pi}{4}-\pi)})\right]$$

$$= \frac{1}{2}\left[X(e^{j\frac{\pi}{4}}) + \underbrace{X(e^{j(-\frac{3\pi}{4})})}_{0}\right]$$

$$= \frac{1}{2}\cdot X(e^{j\frac{\pi}{4}})$$