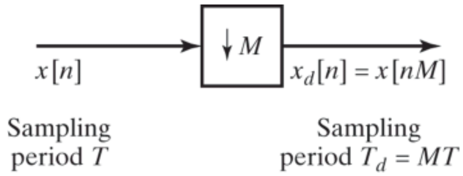
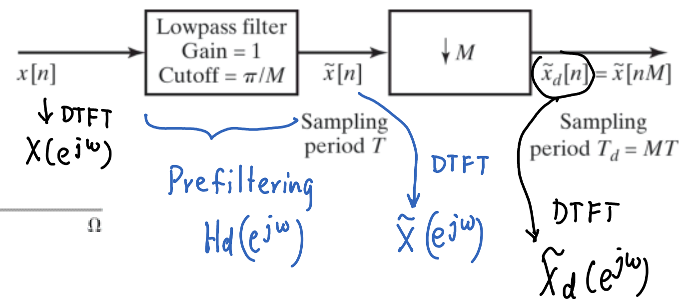
Lecture 12

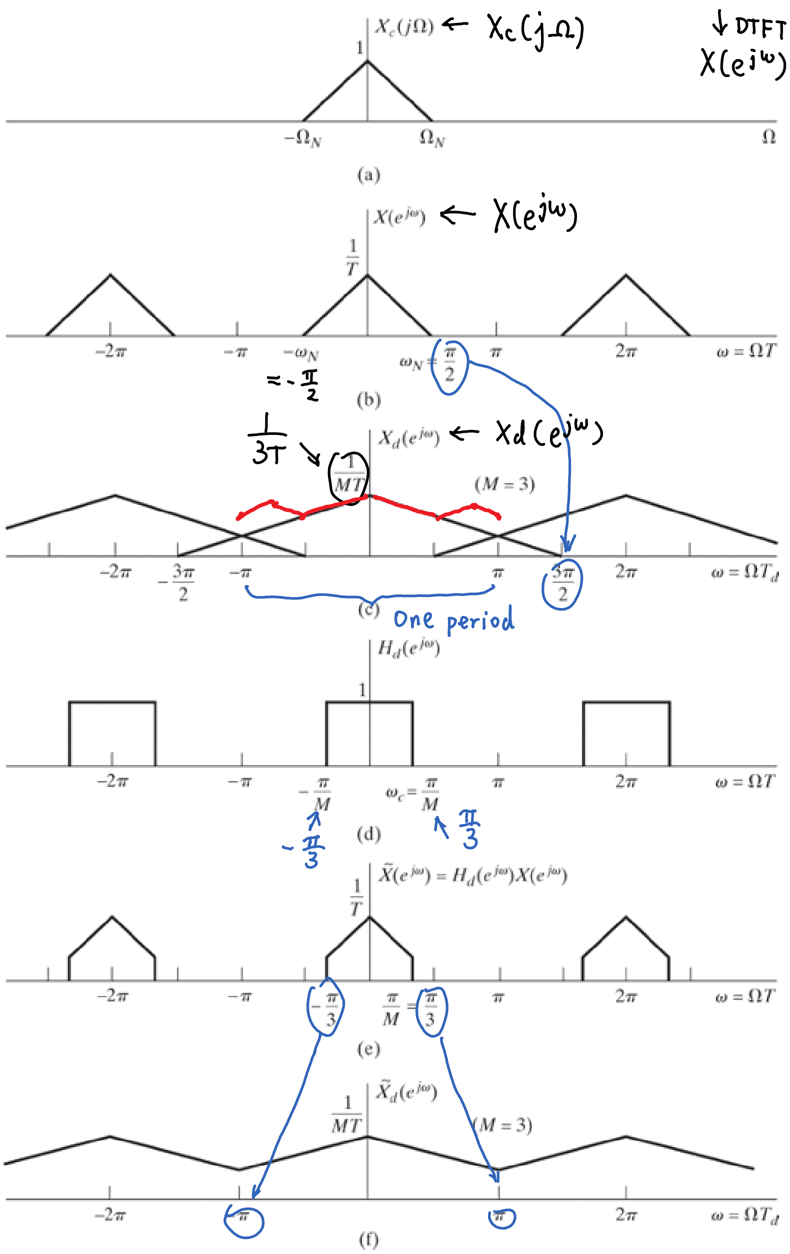
- Down Sampling



$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right)$$

$X_d(e^{j\omega})$: M copies of the DTFT of $x[n]$, $X(e^{j\omega})$ with the frequency $\underbrace{\text{scaled}}_{\text{expansion}}$ by M and shifted by integer multiples of 2π.

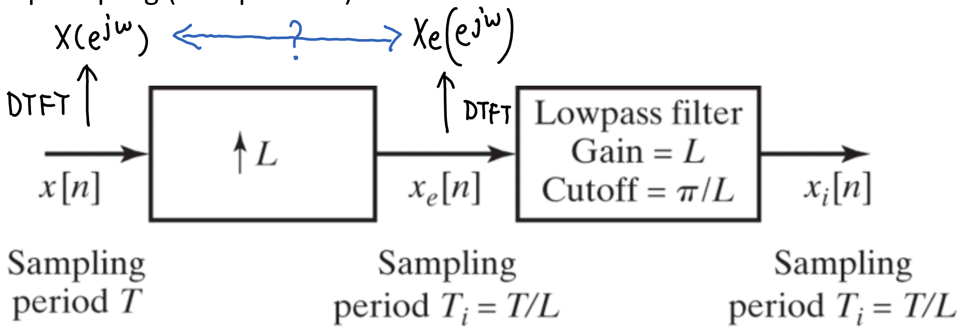




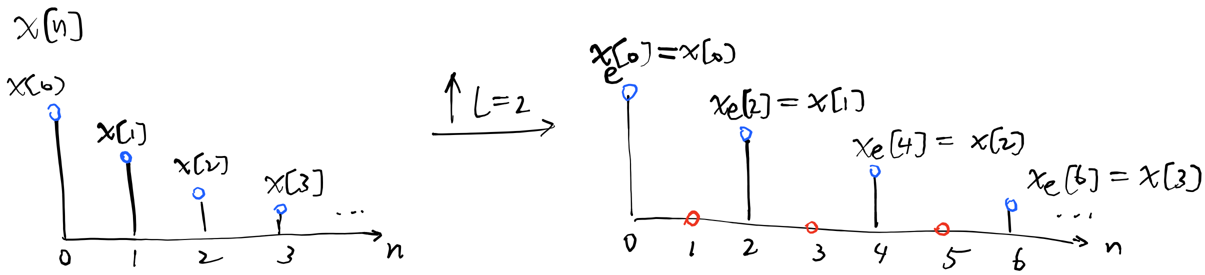
without prefiltering
aliasing occurs

With prefiltering

- Upsampling (Interpolation)



$$x_e[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = 0, \pm L, \pm 2L, \ldots\ldots \\ 0, & \text{otherwise} \end{cases}$$



$$x_e[n] = \begin{cases} x\left[\frac{n}{2}\right], & n = 0, \pm 2, \pm 4, \ldots, \\ 0, & n = \text{odd numbers} \end{cases}$$

Relation between $X(e^{j\omega})$ and $X_e(e^{j\omega})$?

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n - kL]$$

For example, $L = 2$,

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n - 2k],$$

Cases: where $\delta[n-2k] = \begin{cases} 1, & n-2k=0 \\ 0, & n \neq 2k \end{cases}$

(1) If $n = 2k \Rightarrow k = \frac{n}{2}$

$$x_e[n] = \cdots\cdots + x\left[\frac{n}{2}\right] \cdot \underbrace{\delta\left[n - 2 \cdot \frac{n}{2}\right]}_{1} + \cdots\cdots = x\left[\frac{n}{2}\right]$$

(2) If $n \neq 2k \Rightarrow k \neq \frac{n}{2} \Rightarrow n-2k \neq 0 \Rightarrow \delta[n-2k] = 0$

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \underbrace{\delta[n-2k]}_{0} = 0$$

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-kL]$$

$\downarrow$ DTFT

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_e[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-kL] \right\} e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot \left\{ \sum_{n=-\infty}^{n} \delta[n-kL]\, e^{-j\omega n} \right\}$$

where $\delta[n-kL] = \begin{cases} 1, & n = kL \\ 0, & n \neq kL \end{cases}$

$$X_e(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] \cdot e^{-j\omega kL} \underset{\text{Change } k \text{ into } n}{=} \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega L n}$$

In Comparison,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

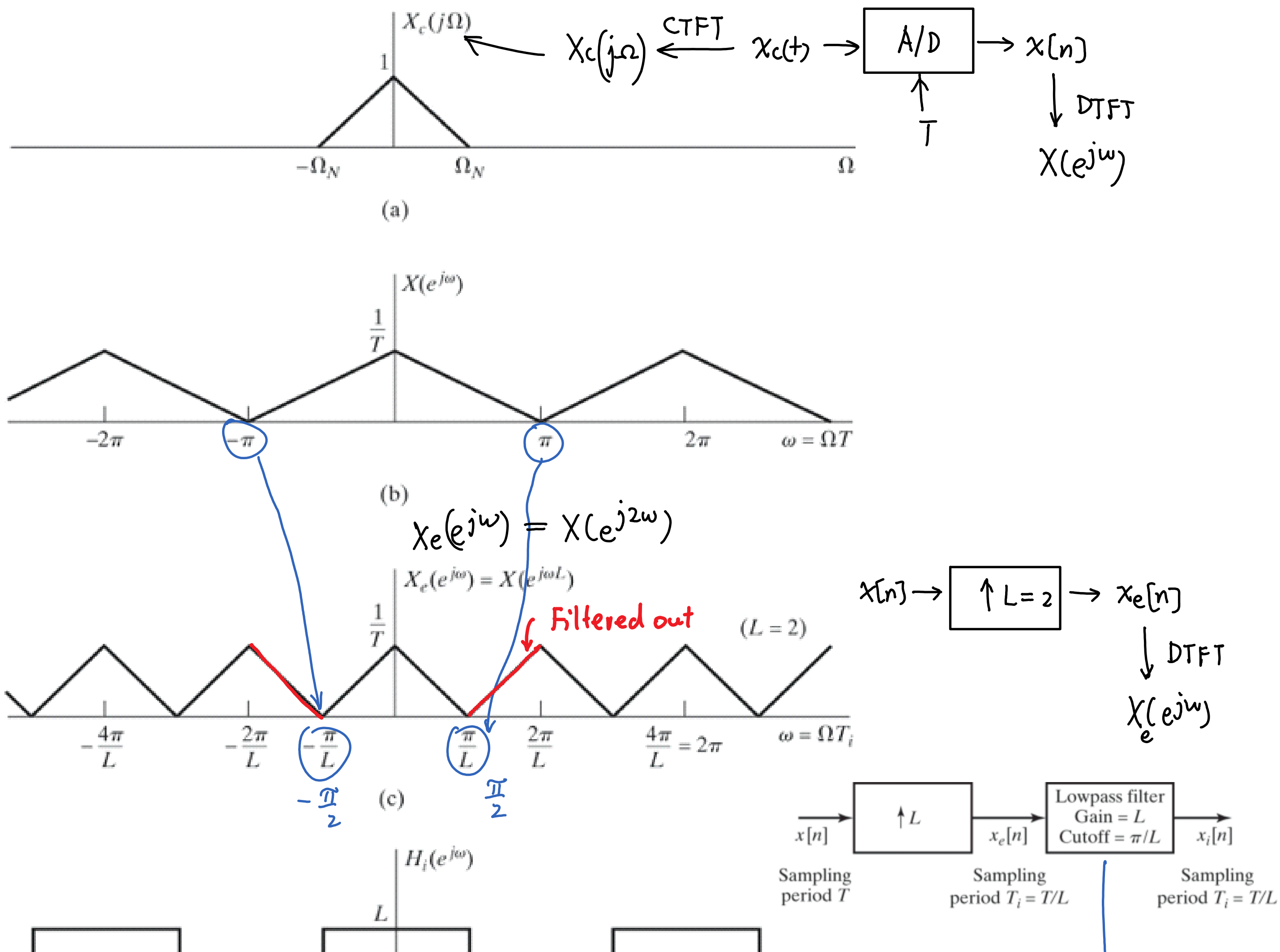
Thus $\boxed{X_e(e^{j\omega}) = X(e^{j\omega L})}$

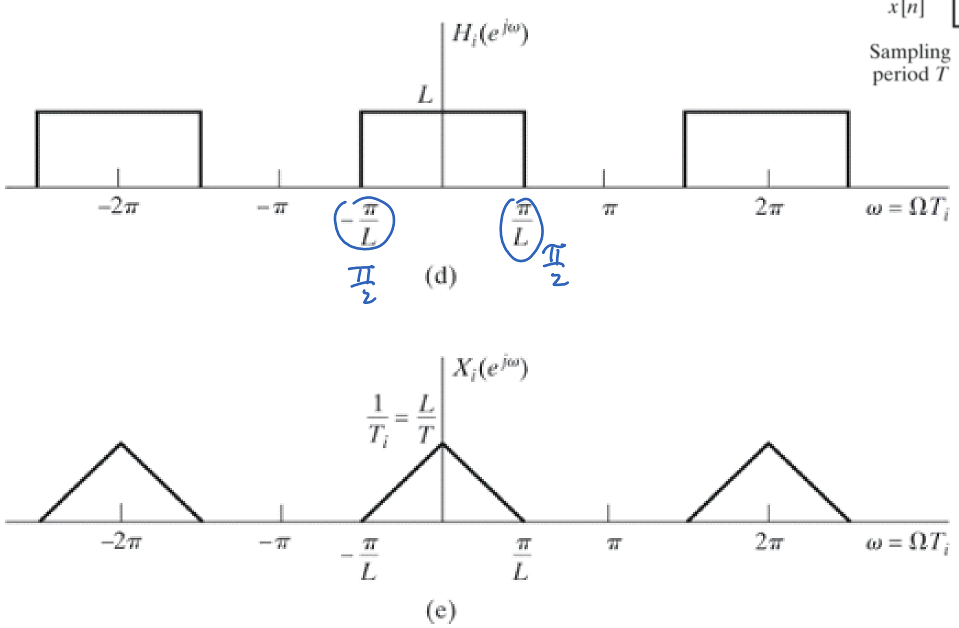
Upsampling in the time domain

$\downarrow$

$X_e(e^{j\omega})$: Scaled (shrinking) Version of the original spectrum $X(e^{j\omega})$

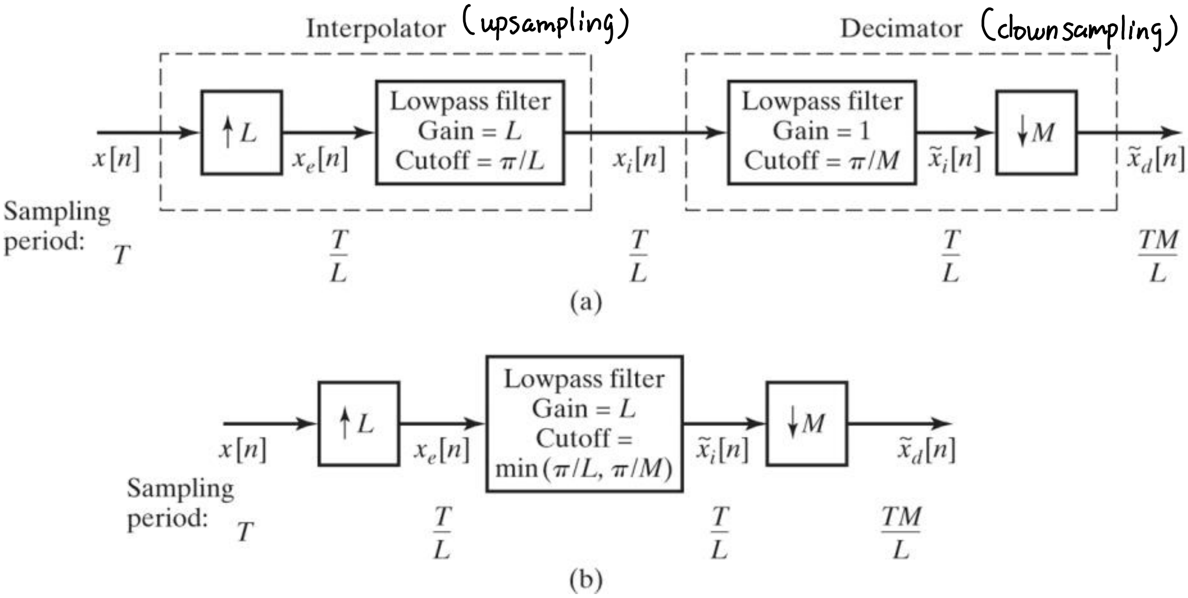
Frequency-Domain Illustration of Interpolation (Upsampling)





Gain = L
to compensate
the high-frequency
components that
are filtered "out"

- Changing the sampling rate by a non-integer factor



In general,

$$x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow y[n]$$

$$\boxed{h_1[n]} \downarrow \; x[n] * h_1[n] \qquad\qquad y[n] \downarrow \; (x[n] * h_1[n]) * h_2[n]$$

In frequency domain

$$X(e^{j\omega}) \rightarrow \boxed{H_1(e^{j\omega})} \rightarrow \boxed{H_2(e^{j\omega})} \rightarrow Y(e^{j\omega})$$

$$X(e^{j\omega}) \cdot H_1(e^{j\omega}) \qquad\qquad \left[X(e^{j\omega}) \cdot H_1(e^{j\omega})\right] \cdot H_2(e^{j\omega})$$

$$= X(e^{j\omega}) \cdot \left[H_1(e^{j\omega}) \cdot H_2(e^{j\omega})\right]$$

⇓ Equivalent System

$$X(e^{j\omega}) \rightarrow \boxed{H_1(e^{j\omega}) \cdot H_2(e^{j\omega})} \rightarrow Y(e^{j\omega})$$

$$x_c(t) \to \boxed{A/D} \xrightarrow{x[n]} \boxed{\uparrow L=2} \to \boxed{\begin{matrix}\text{LPF}\\ \text{Gain} = L = 2\\ \text{Cutoff} = \frac{\pi}{3}\end{matrix}} \to \boxed{\downarrow M=3} \to \tilde{x}_d[n]$$

$x_c(t) \xrightarrow{\text{CTFT}} X_c(j\Omega)$ $\quad$ A/D $\uparrow$ T $\quad$ $x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega})$

$$\min\left(\frac{\pi}{L}, \frac{\pi}{M}\right) = \min\left(\frac{\pi}{2}, \frac{\pi}{3}\right) = \frac{\pi}{3}$$

$$T \to T \cdot \frac{M}{L} = T \cdot \frac{3}{2}$$

$\tilde{x}_d[n]$: $\frac{2}{3}$ of the # of samples in $x[n]$