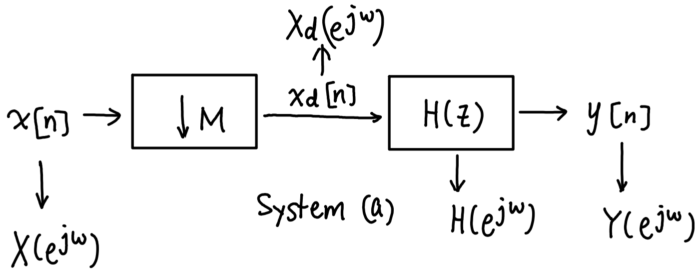
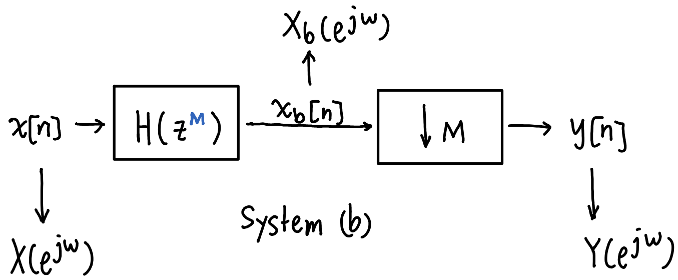
Lecture 13

Multirate Signal Processing

- Interchange of filtering and compressor





From System (b):

$$X_b(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{jM\omega}) \qquad \cdots\cdots (1)$$

From System (a): in Lecture 11

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n]$$

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right)$$

Similarly in System (b)

$$x_b[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y[n]$$

$$Y(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X_b\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right) \qquad \cdots\cdots (2)$$

Plugging (1) into (2):

$$Y(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right) \underbrace{H\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)M}\right)}_{H\left(e^{j(\omega - 2\pi i)}\right)}$$

$$= H\left(e^{j\omega} \cdot \underbrace{e^{-j2\pi i}}_{1}\right) = H(e^{j\omega})$$

Thus

$$Y(e^{j\omega}) = H(e^{j\omega}) \cdot \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right)$$

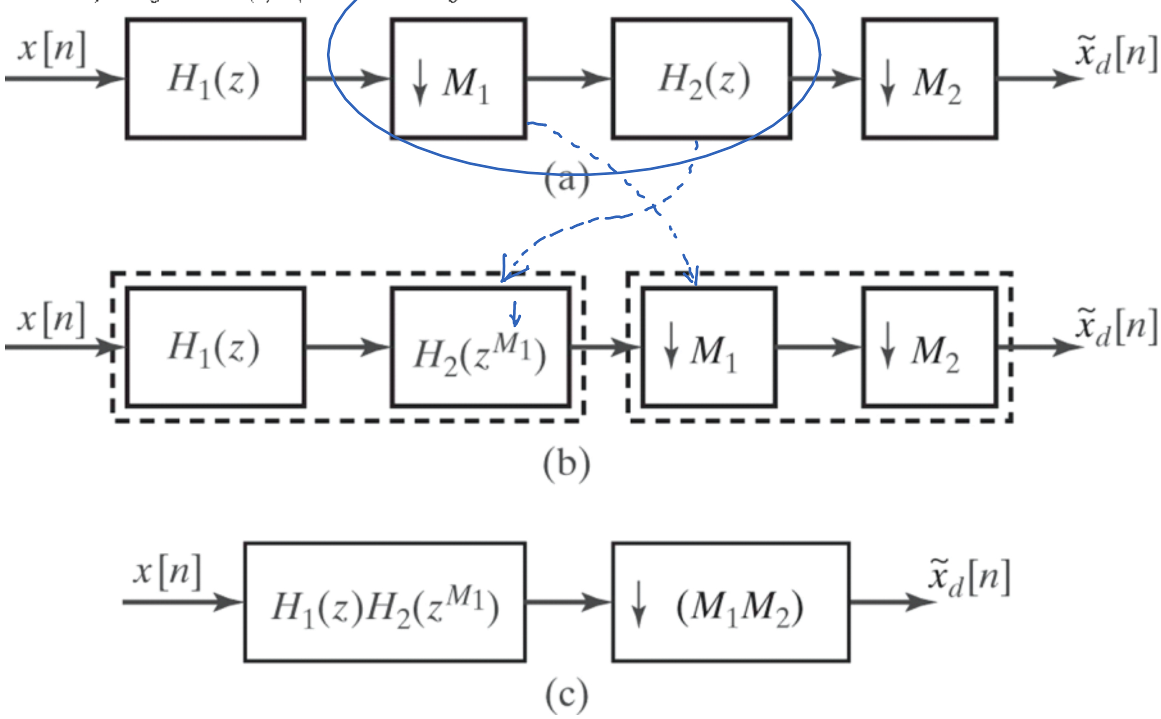
But

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right)$$

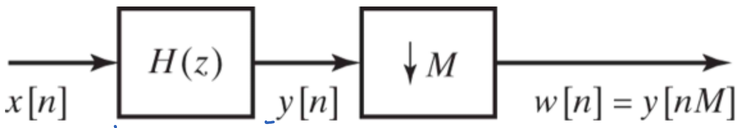
$$Y(e^{j\omega}) = H(e^{j\omega}) \cdot X_d(e^{j\omega}) \Rightarrow \text{System (a)}$$

Therefore, Systems (a) and (b) are equivalent. Down Sampling Identity.

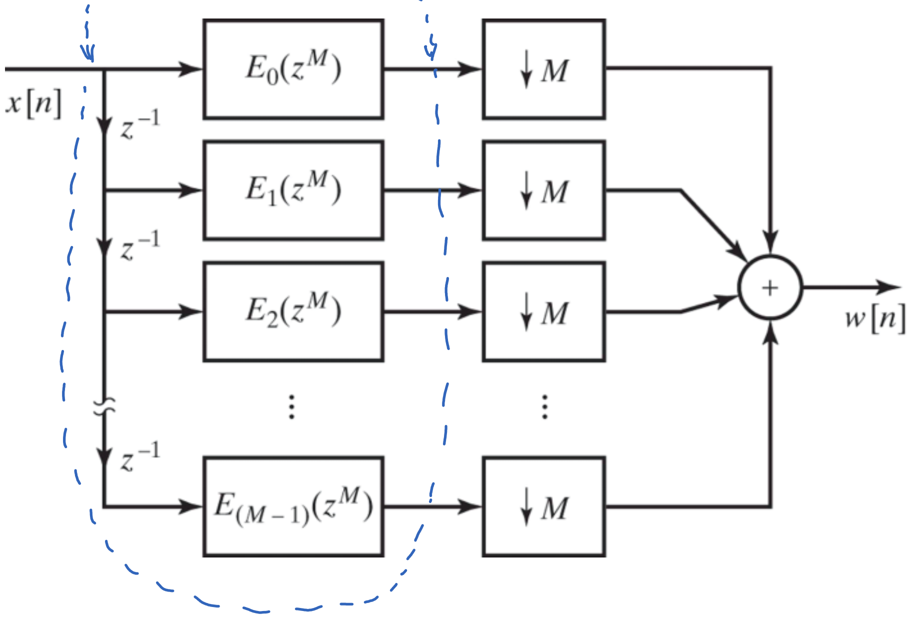
Figure 4.33 Multistage decimation: (a) Two-stage decimation system. (b) Modification of (a) using downsampling identity of Figure 4.31. (c) Equivalent one-stage decimation.



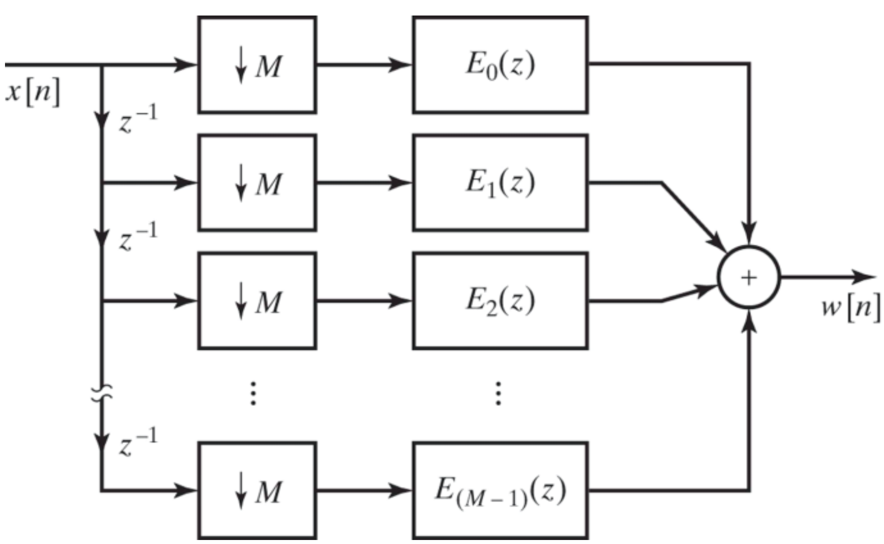
- Polyphase Decomposition



More efficient implementation of the above system:



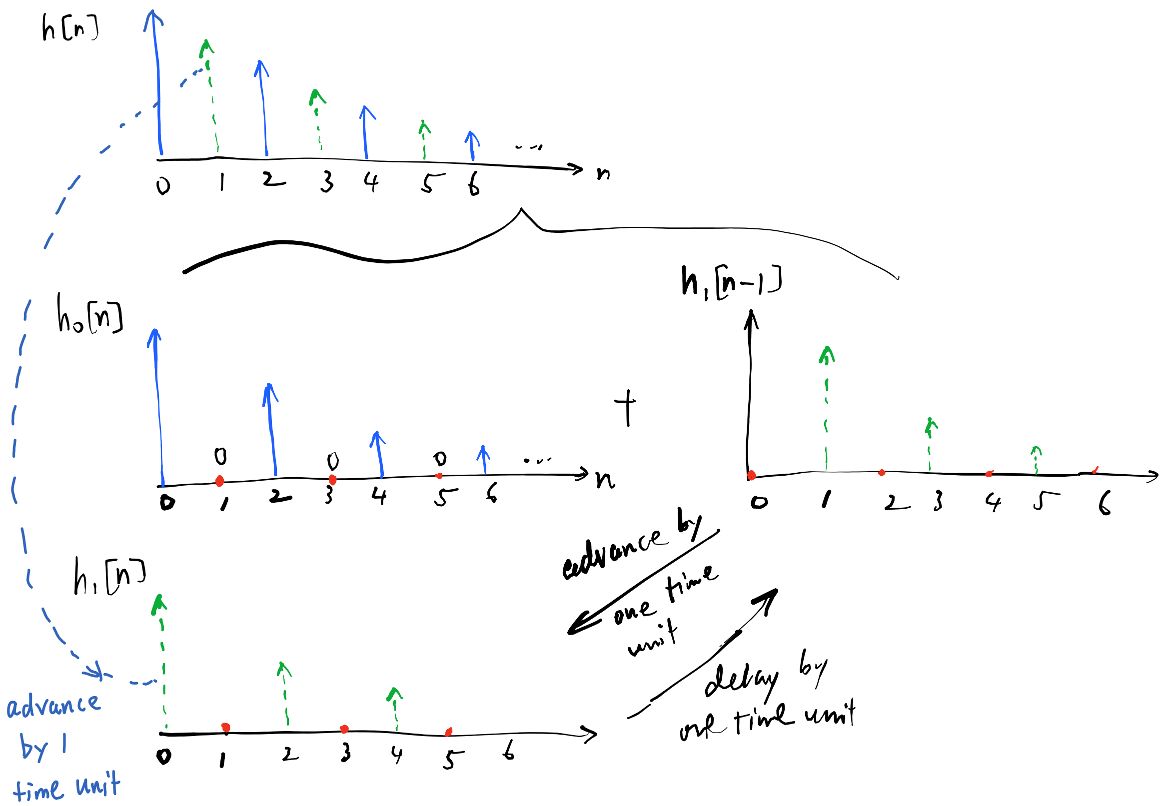
↓ Apply downsampling identity



Polyphase Decomposition

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k] = h_0[n] + h_1[n-1] + \cdots + h_{M-1}[n-(M-1)]$$

For example $M=2$, then $h[n] = \sum_{k=0}^{2-1} h_k[n-k] = h_0[n] + h_1[n-1]$



In general, $e_k[n] = h_k[nM]$

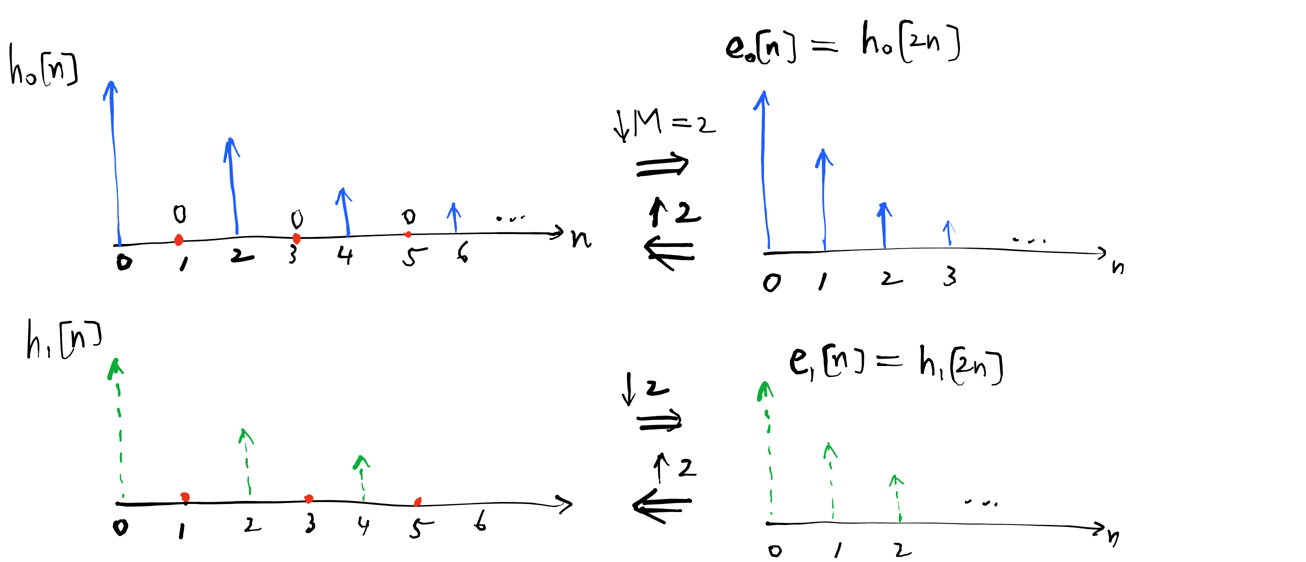


Figure 4.35 Polyphase decomposition of filter $h[n]$ using components $e_k[n]$.

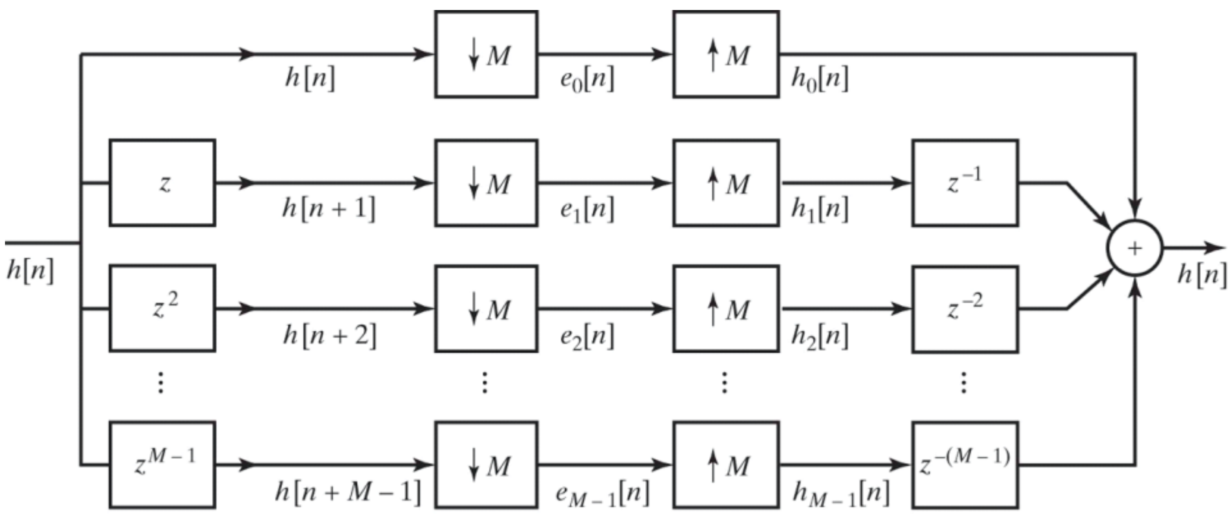
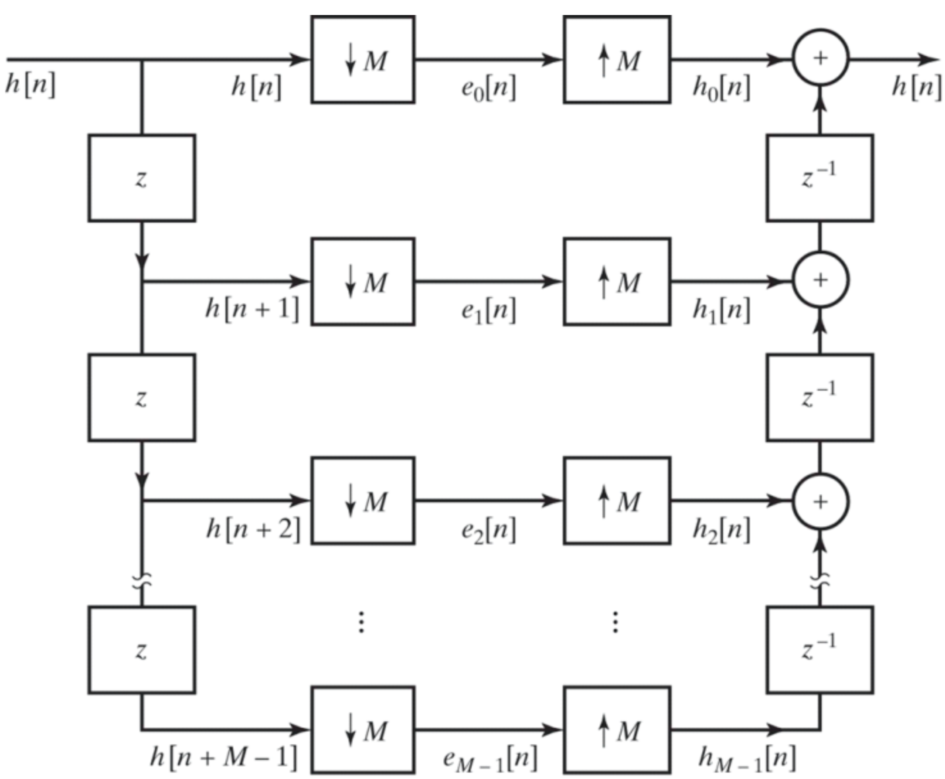


Figure 4.36 Polyphase decomposition of filter $h[n]$ using components $e_k[n]$ with chained delays.



$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

$\downarrow$ ZT

$$H(z) = \sum_{k=0}^{M-1} Z\{h_k[n-k]\} = \sum_{k=0}^{M-1} H_k(z)\cdot z^{-k} \quad \ldots\ldots (1)$$

where $h_k[n] \xrightarrow{ZT} H_k(z)$

$h_k[n-k] \xrightarrow{ZT} H_k(z)\cdot z^{-k}$ Time-Shift Property of Z-T

Now, $e_k[n] = h_k[nM]$

$\downarrow$ ZT

$$E_k(z) = \sum_{n=-\infty}^{\infty} e_k[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} h_k[\underbrace{nM}_{s}]\cdot z^{-n}$$

Thus $E_k(z^M) = \sum_{n=-\infty}^{\infty} h_k[\underbrace{nM}_{s}]\cdot z^{-\underbrace{Mn}_{s}}$

Let $s = nM$, $E_k(z^M) = \sum_{s=-\infty}^{\infty} h_k[s]\cdot z^{-s} = H_k(z)$

Therefore $E_k(z^M) = H_k(z)$

By Eq. (1) : $H(z) = \sum_{k=0}^{M-1} H_k(z)\cdot z^{-k}$

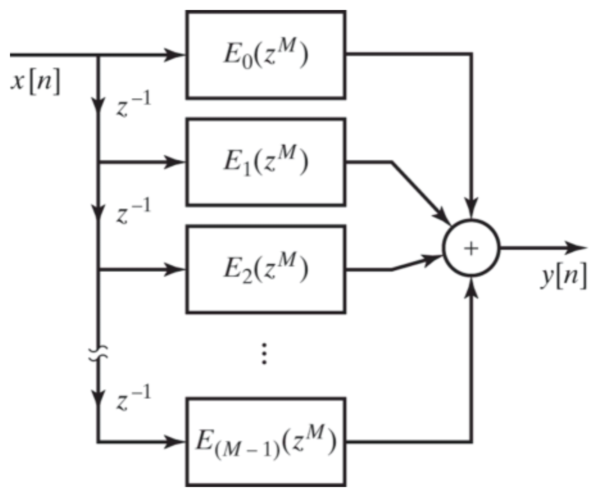
Hence $H(z) = \sum_{k=0}^{M-1} E_k(z^M)\cdot z^{-k}$

$$= E_0(z^M) + E_1(z^M)\cdot z^{-1} + E_2(z^M)\cdot z^{-2} + \cdots E_{M-1}(z^M)\cdot z^{-(M-1)}$$

$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) \cdot z^{-k}$$

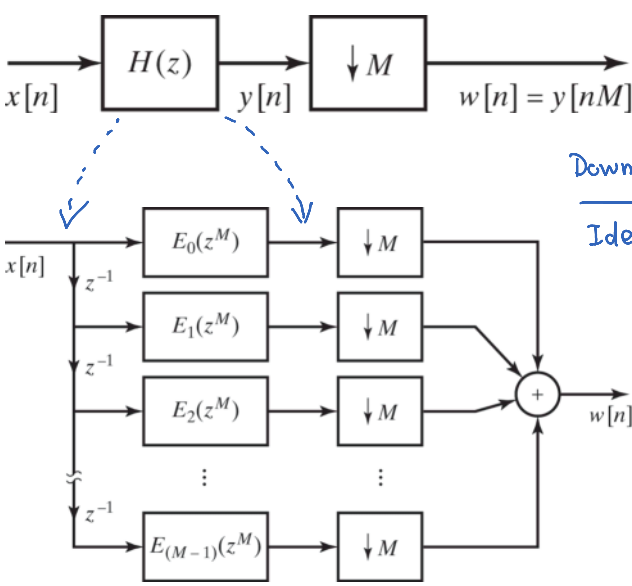
$$= E_0(z^M) + E_1(z^M) \cdot z^{-1} + E_2(z^M) \cdot z^{-2} + \cdots E_{M-1}(z^M) \cdot z^{-(M-1)}$$

Figure 4.37 Realization structure based on polyphase decomposition of $h[n]$.



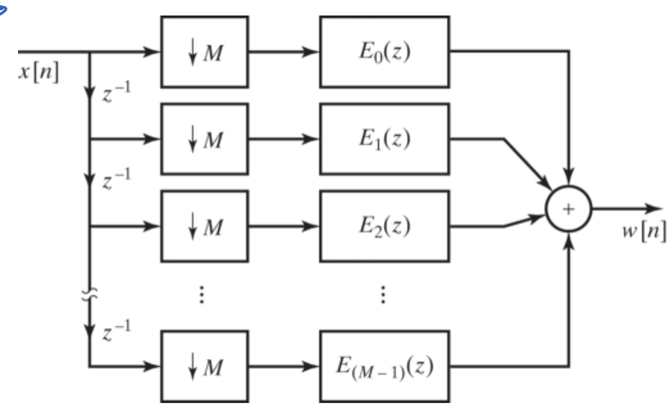
Equivalent to:

$$x[n] \rightarrow \boxed{H(z)} \rightarrow y[n]$$



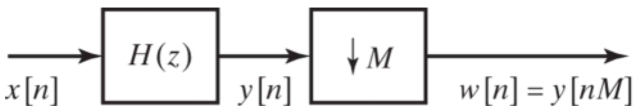
Downsampling

Identity



Comparison of computational complexity

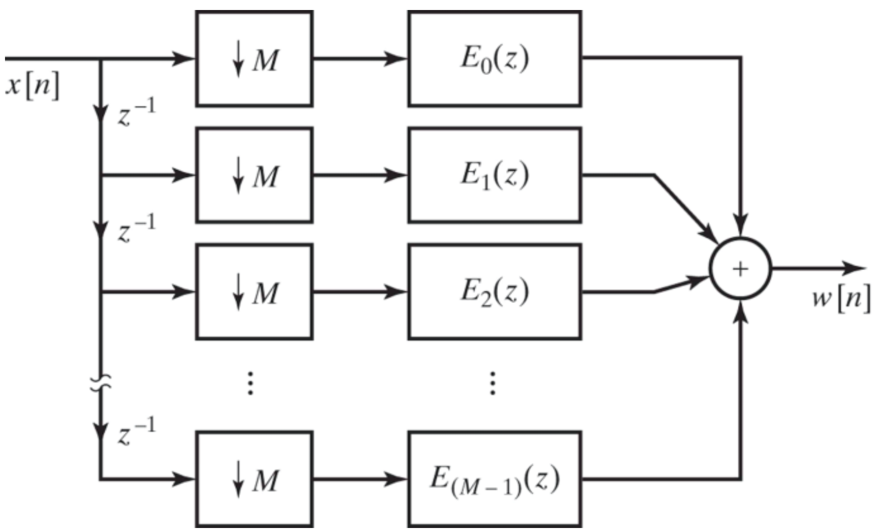
Direct Implementation



Input x[n] is clocked at a rate of one sample per unit time and H(z) N-point FIR filter

\# of Multiplications (mults): N
\# of Additions (adds): (N-1)

Implementation of the decimation filter with polyphase decomposition



Each of the filter $E_k(z)$ is of length N/M, and their inputs are clocked at a rate of 1 per M units of time.

\# of Multiplications (mults): (1/M)x(N/M)
\# of Additions (adds): (1/M)x(N/M - 1)

Since there are M polyphase filters, so the total

\# of Multiplications (mults): M x (1/M)x(N/M) = N/M $\ll N$
\# of Additions (adds): M x (1/M)x(N/M - 1) + (1/M)x(M-1) = $\frac{N}{M} - 1 + \frac{M-1}{M} \ll N-1$