Lecture 14

Chapter 5 Transform Analysis of LTI Systems

- Frequency Response of LTI Systems

$$\begin{tikzcd} x[n] \arrow[r] \arrow[d] & \boxed{h[n]} \arrow[r] \arrow[d] & y[n] \arrow[d] \\ X(z) & H(z) & Y(z) \end{tikzcd}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

In Z-Transform domain $\qquad Y(z) = X(z) \cdot \underbrace{H(z)}_{\text{System Function}}$

In Fourier-Transform domain

$$Y(e^{jw}) = X(e^{jw}) \cdot H(e^{jw})$$

Magnitude Response： $\qquad |H(e^{jw})|$: gain of the system

$$|Y(e^{jw})| = |X(e^{jw})| \cdot |H(e^{jw})|$$

Phase Response: $\qquad \angle H(e^{jw})$: Phase shift of the system

$$Y(e^{jw}) = |Y(e^{jw})| \cdot e^{j\angle Y(e^{jw})}$$

$$Y(e^{jw}) = X(e^{jw}) \cdot H(e^{jw})$$

$$|Y(e^{jw})| e^{j\angle Y(e^{jw})} = |X(e^{jw})| e^{j\angle X(e^{jw})} \cdot |H(e^{jw})| e^{j\angle H(e^{jw})}$$

$$= |X(e^{jw}) \cdot H(e^{jw})| \, e^{j[\angle X(e^{jw}) + \angle H(e^{jw})]}$$

Thus, $\quad |Y(e^{jw})| = |X(e^{jw}) \cdot H(e^{jw})|$

and $\quad \angle Y(e^{jw}) = \angle X(e^{jw}) + \underbrace{\angle H(e^{jw})}_{\text{Phase shift}}$

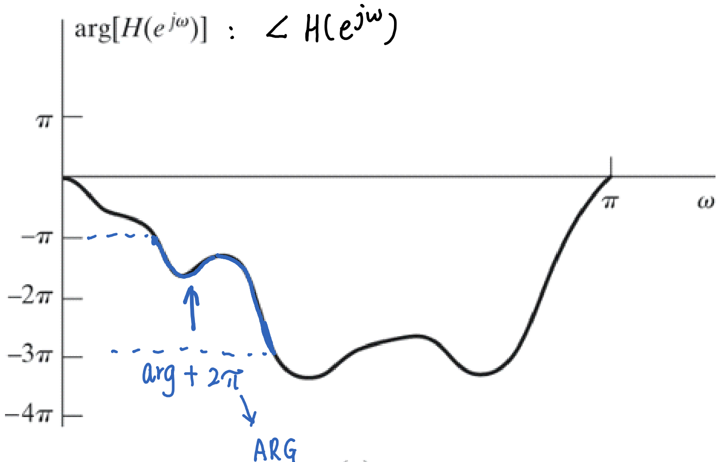
- Principal value of the Phase Response : $\quad ARG[H(e^{jw})]$

$$\angle H(e^{jw}) = ARG[H(e^{jw})] + 2\pi \cdot r(w),$$

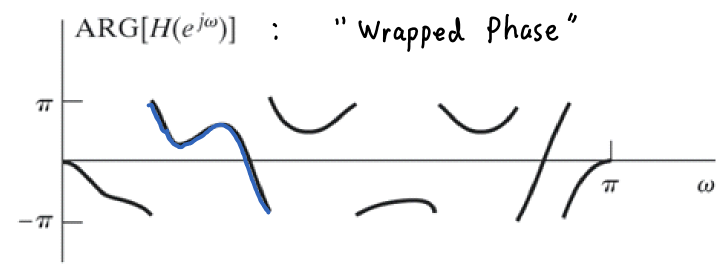
where $r(w)$ is a positive or negative integer

$$-\pi < ARG[H(e^{jw})] \leq \pi$$

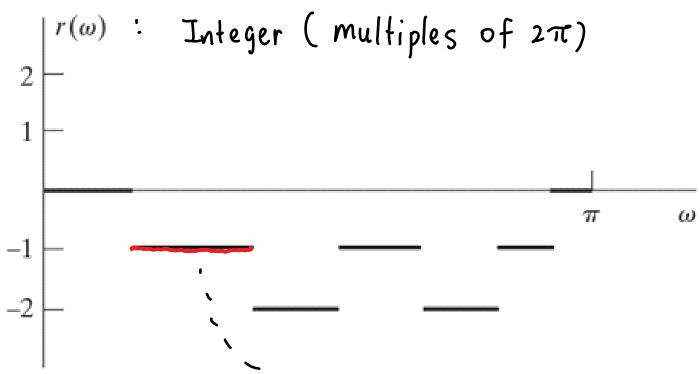
$r(w)$ can be different at each w.



(a)



(b)



(c)

$$\underbrace{\angle H(e^{j\omega})}_{\text{Subplot (a)}} = \underbrace{ARG[H(e^{j\omega})]}_{\text{Subplot (b)}} + 2\pi \cdot \underbrace{\underbrace{r(\omega)}_{-1}}_{\text{Subplot (c)}}$$

- Group Delay of a System

$$\tau(\omega) \triangleq \text{grd}\left[H(e^{j\omega})\right] = -\frac{d}{d\omega}\left\{\arg\left[H(e^{j\omega})\right]\right\}$$

Example: Ideal Delay System

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n-n_d]$$

$$\downarrow ?$$

$$\text{If } x[n] = \delta[n], \quad h[n] = \delta[n-n_d]$$

$$H(e^{j\omega}) = \text{DTFT}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n] \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n-n_d]\, e^{-j\omega n}$$

$$= e^{-j\omega n_d}$$

$$H(z) = ZT\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n] \cdot z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n-n_d]\, z^{-n}$$

$$= z^{-n_d}$$

$$H(e^{j\omega}) = e^{-j\omega n_d} \Rightarrow \angle H(e^{j\omega}) = -\omega n_d$$

$$\text{grd}\left[H(e^{j\omega})\right] = -\frac{d}{d\omega}(-\omega n_d) = n_d$$

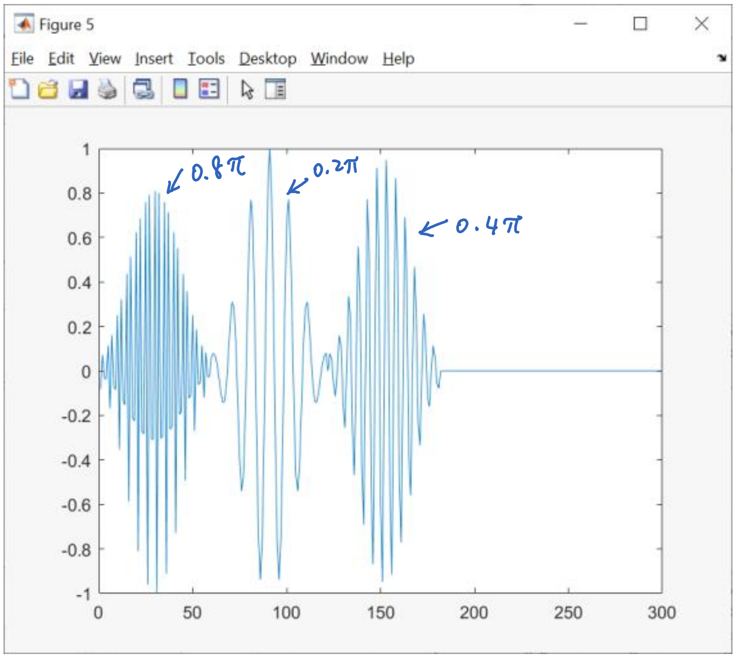
Another Example (more complicated system):

$$H(z) = \underbrace{\left(\frac{(1-.98e^{j.8\pi}z^{-1})(1-.98e^{-j.8\pi}z^{-1})}{(1-.8e^{j.4\pi}z^{-1})(1-.8e^{-j.4\pi}z^{-1})}\right)}_{H_1(z)} \underbrace{\prod_{k=1}^{4}\left(\frac{(c_k^* - z^{-1})(c_k - z^{-1})}{(1-c_k z^{-1})(1-c_k^* z^{-1})}\right)^2}_{H_2(z)}$$

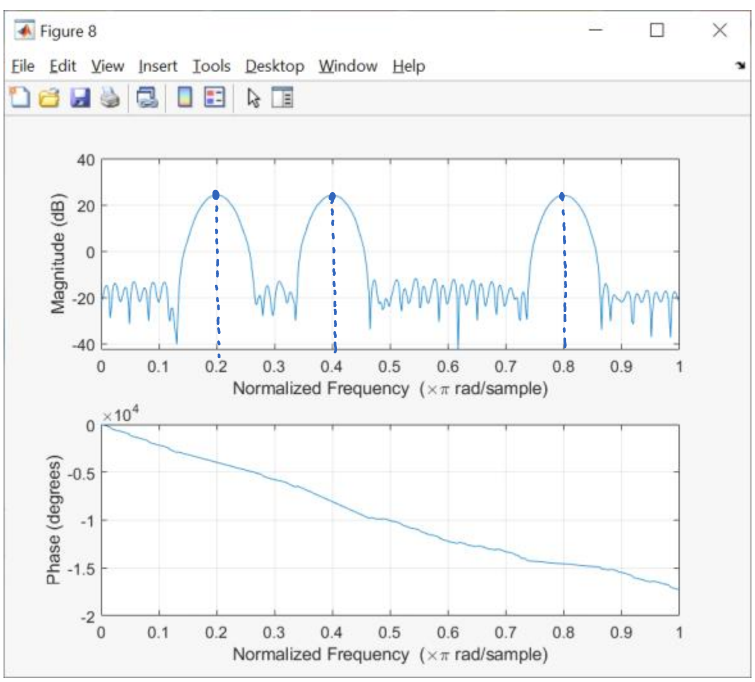
with $c_k = 0.95e^{j(.15\pi+.02\pi k)}$ for $k = 1, 2, 3, 4$ and $H_1(z)$ and $H_2(z)$ defined

Illustration of system effects of Group Delay and Attenuation http://www.ece.uah.edu/~dwpan/course/ee648/code/demo.m

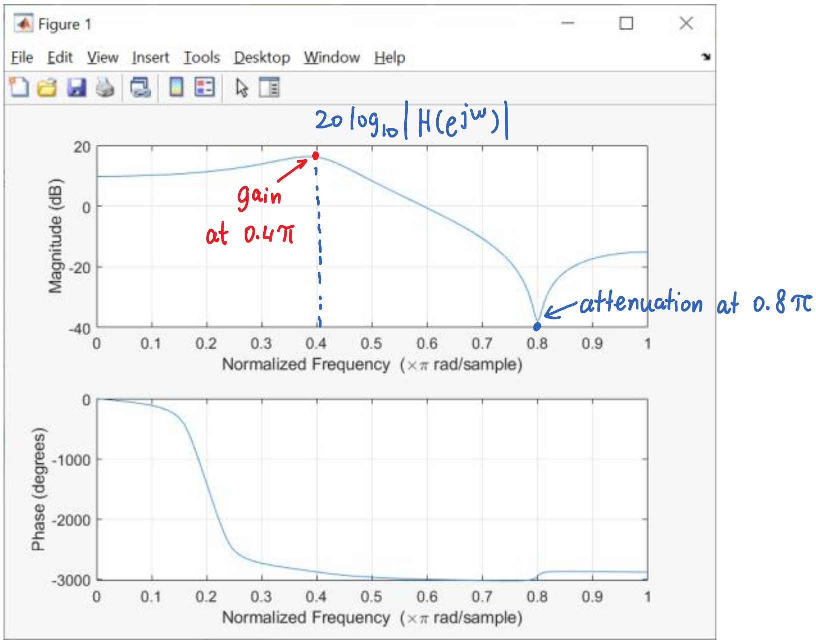
x[n]



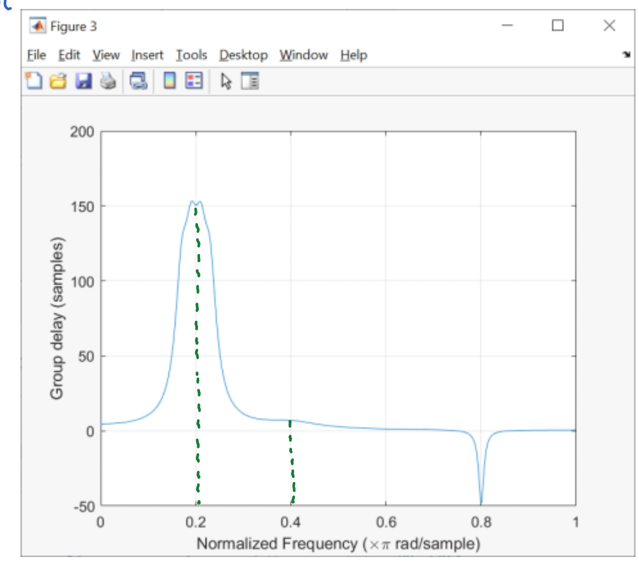
figure; freqz(x,1)



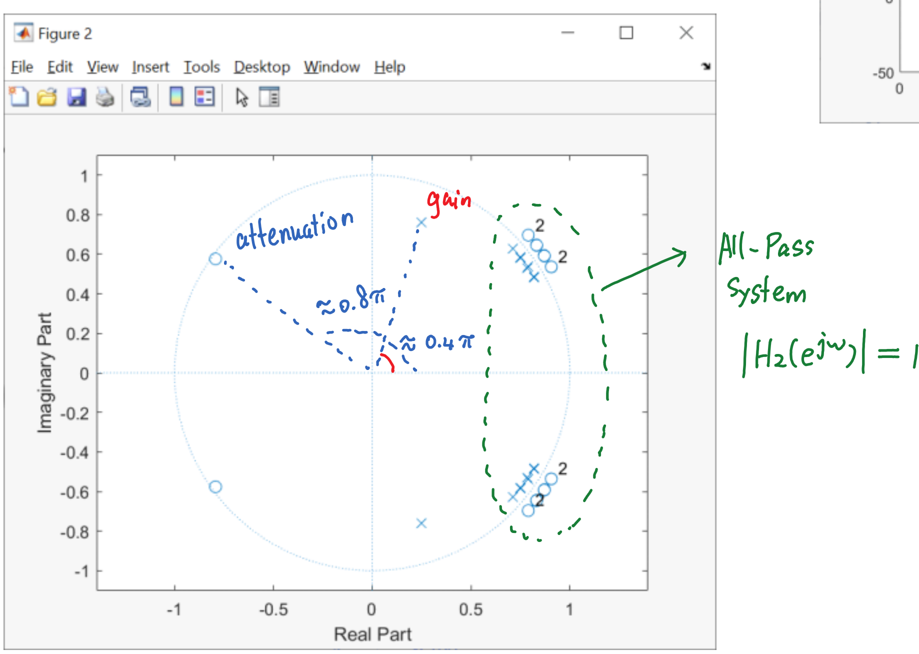
System:



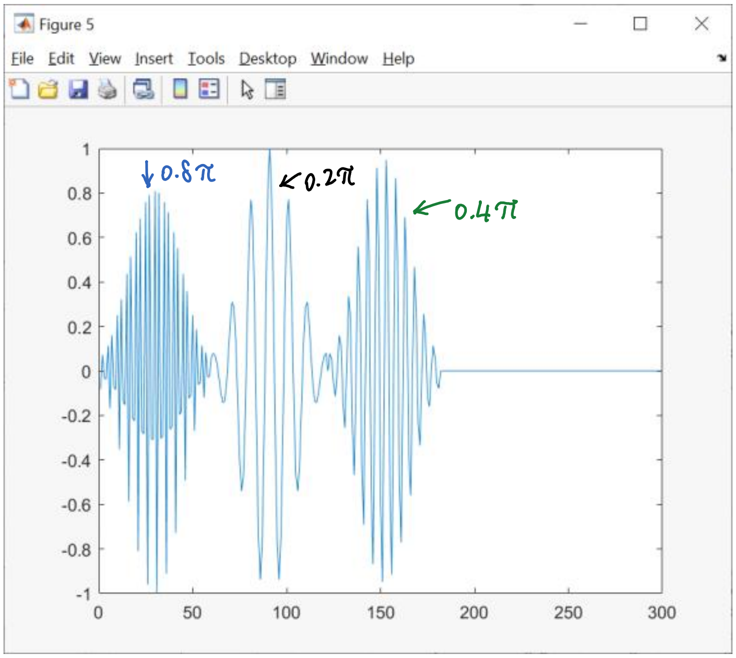
Group Delay



Zero-pole plot

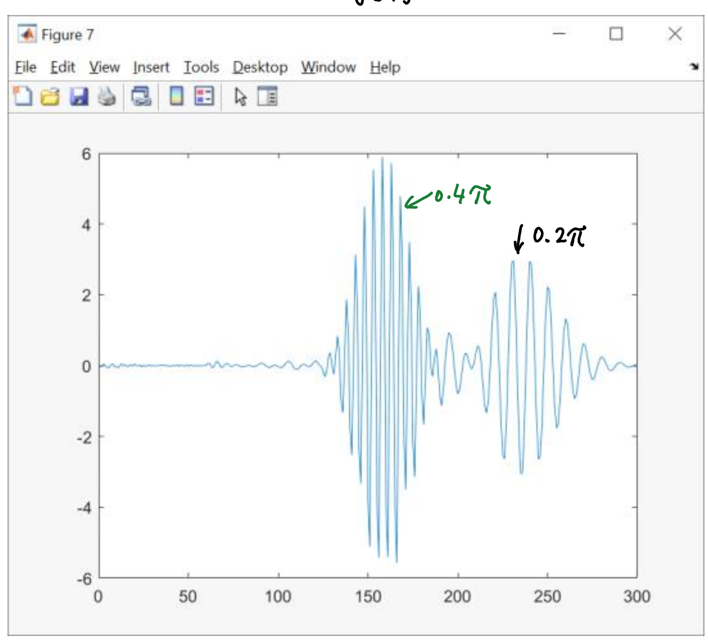


x[n]



$\boxed{H(z)}$

y[n]



Effects of $\underbrace{\text{Group Delay}}_{grd[H(e^{j\omega})]}$ and $\underbrace{\text{Attenuation}}_{|H(e^{j\omega})|}$

$\omega =$

- 0.8π component : attenuated
- 0.2π component : Some gain and delayed by ≈ 150 samples
- 0.4π component : largest gain and not much delay

- Linear constant-coefficient difference equations

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

c_k : zero's

d_k : poles

Example:

$$H(z) = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2} z^{-1})(1 + \frac{3}{4} z^{-1})} = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4} z^{-1} - \frac{3}{8} z^{-2}} = \frac{Y(z)}{X(z)}$$

Input-Output Relation:

$$(1 + 2z^{-1} + z^{-2}) X(z) = (1 + \tfrac{1}{4} z^{-1} - \tfrac{3}{8} z^{-2}) Y(z)$$

$$x[n] + 2x[n-1] + x[n-2] = y[n] + \tfrac{1}{4} y[n-1] - \tfrac{3}{8} y[n-2]$$

Poles and Zeros

$(1 + z^{-1}) = 0 \Rightarrow z = -1$ Two zeros

$(1 - \frac{1}{2} z^{-1})(1 + \frac{3}{4} z^{-1}) = 0 \Rightarrow z = \frac{1}{2}, \quad z = -\frac{3}{4}$ Two poles

$$(1 + z^{-1})^2 = [1 - \underbrace{(-1)}_{c_1} z^{-1}][1 - \underbrace{(-1)}_{c_2} z^{-1}]$$

$$(1 - \tfrac{1}{2} z^{-1})(1 + \tfrac{3}{4} z^{-1}) = (1 - \underbrace{\tfrac{1}{2}}_{d_1} z^{-1})[1 - \underbrace{(-\tfrac{3}{4})}_{d_2} z^{-1}]$$

- Stability and Causality
  Check based on ROC of H(z)

Example:

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]$$

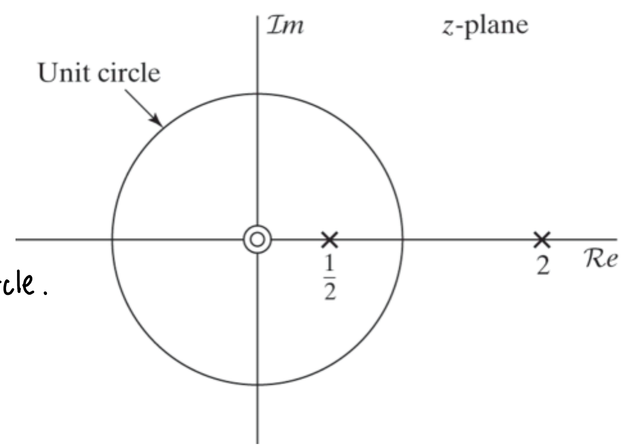
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$Y(z) - \frac{5}{2}z^{-1}Y(z) + z^{-2}Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{5}{2}z^{-1}+z^{-2}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} \quad \begin{matrix} \times z^2 \\ \times z \cdot z \end{matrix}$$

$$= \frac{z^2}{(z-\frac{1}{2})(z-2)}$$

ROC
several cases:



(1) $|z| > 2$

Causal : Yes

Stable : No

ROC does not include the unit circle.

(2) $\frac{1}{2} < |z| < 2$

Causal : No

Stable : Yes

(3) $|z| < \frac{1}{2}$

Causal : No

Stable : No