Lecture 15

- Inverse Systems

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow \boxed{H_i(z)} \longrightarrow y[n] = x[n]$$

$$G(z) = H(z) \cdot H_i(z) = 1$$

$H_i(z)$: Inverse System of $H(z)$

$$H_i(z) = \frac{1}{H(z)}, \quad H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

$$G(z) = 1 \xrightarrow{\mathcal{Z}^{-1}} g[n] = \delta[n]$$

If $$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})},$$ Thus zeros : c_k

poles : d_k

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N} (1 - d_k z^{-1})}{\prod_{k=1}^{M} (1 - c_k z^{-1})},$$ Thus zeros : d_k

poles : c_k

Examples

(1)

$$H(z) = \frac{1 - 0.5 z^{-1}}{1 - 0.9 z^{-1}},$$ RoC : $|z| > 0.9$

$$\begin{cases} \text{zero}: & 0.5 \\ \text{pole}: & 0.9 \end{cases}$$

$$\frac{1 - 0.5 z^{-1}}{1 - 0.9 z^{-1}} \cdot \frac{z}{z} = \frac{z - 0.5}{z - 0.9}$$

$$\begin{cases} z - 0.5 = 0 \Rightarrow z = 0.5 \text{ (zero)} \\ z - 0.9 = 0 \Rightarrow z = 0.9 \text{ (pole)} \end{cases}$$

$$H_i(z) = \frac{1 - 0.9 z^{-1}}{\underbrace{1 - 0.5 z^{-1}}_{\text{pole}:\ 0.5}},$$ RoC ? $\left.\begin{matrix} |z| > 0.9 \\ |z| > 0.5 \end{matrix}\right\} \Rightarrow |z| > 0.9$

Two possible cases

$$\begin{cases} |z| > 0.5 \ \checkmark \\ |z| < 0.5 : \text{ No overlap with } |z| > 0.9 \end{cases}$$

How about $h_i[n]$?

$$H_i(z) = \frac{1-0.9z^{-1}}{1-0.5z^{-1}}, \quad \text{ROC}: |z| > 0.9$$

$$= \underbrace{\frac{1}{1-0.5z^{-1}}}_{\downarrow} - \underbrace{\frac{0.9z^{-1}}{1-0.5z^{-1}}}_{\downarrow}, \quad \text{where} \quad \frac{1}{1-0.5z^{-1}} \xrightarrow{z^{-1}} (0.5)^n u[n]$$

$$\frac{z^{-1}}{1-0.5z^{-1}} \xrightarrow{z^{-1}} (0.5)^{n-1} u[n-1]$$

$$h_i[n] = (0.5)^n u[n] - 0.9 \cdot (0.5)^{n-1} u[n-1]$$

Time-Shift Property of the Z-T.

(2) $$H(z) = \frac{z^{-1}-0.5}{1-0.9z^{-1}}, \quad |z| > 0.9$$

$$= \frac{-0.5(1-2z^{-1})}{1-0.9z^{-1}} \Rightarrow \begin{array}{ll} \text{zero}: & 2 \\ \text{pole}: & 0.9 \end{array}$$

$$H_i(z) = \frac{1-0.9z^{-1}}{-0.5(1-2z^{-1})} \Rightarrow \begin{array}{ll} \text{zero}: & 0.9 \\ \text{pole}: & 2 \end{array}$$

$\downarrow$

$h_i[n]$

$$\text{ROC}: \begin{cases} |z| > 2 & \cdots \text{ Case 1} \\ |z| < 2 & \cdots \text{ Case 2} \end{cases}$$

$$H_i(z) = \frac{-2(1-0.9z^{-1})}{1-2z^{-1}}$$

$$= \frac{-2}{1-2z^{-1}} + \frac{1.8z^{-1}}{1-2z^{-1}}$$

Case 1: $|z| > 2$ (intersection of $|z| > 0.9$ and $|z| > 2$)

$$h_i[n] = -2 \cdot 2^n u[n] + 1.8 \, 2^{n-1} u[n-1]$$

System $H_i(z)$ is non-stable (since the ROC excludes the unit circle) and causal (ROC is going outward)

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8:28 AM

Case 2 : $0.9 < |z| < 2$ (Ring)

System $H_i(z)$ is stable and noncausal

$$H_i(z) = \frac{-2}{1-2z^{-1}} + \frac{1.8\, z^{-1}}{1-2z^{-1}}$$

$$\frac{1}{1-az^{-1}} \xrightarrow{Z^{-1}} \begin{cases} a^n u[n], & \text{if } |z|>|a| \\ -a^n u[-n-1], & \text{if } |z|<|a| \end{cases}$$

$$h_i[n] = (-2)\left\{-2^n u[-n-1]\right\} + 1.8\left\{-2^{n-1} u[-(n-1)-1]\right\}$$

$$= 2^{n+1} u[-n-1] - 1.8 \cdot 2^{n-1} u[-n]$$: left-sided sequence $\rightarrow$ noncausal

$n \rightarrow -\infty$, $h_i[n] \rightarrow 0$: stable

- Minimum-Phase System

An LTI system that is both stable and causal, with all poles d_k and zeros c_k inside the unit circle.

ROC: $|z| > \max_{k=1,2,\dots,M} |c_k|$ going outward

A minimum-phase system has an inverse system that is also stable and causal.

If $H(z)$ is a minimun - phase system (causal and stable)

with zeros : c_k, $k = 1, 2, \dots, M$ (inside the unit circle)

poles of the inverse system $H_i(z)$: c_k's are inside the unit circle

Two possible cases of the ROC of $H_i(z)$

$\underbrace{\text{Case 1}}_{\text{of interest}}$: ROC of $H_i(z)$: $|z| > \max_{k=1,2,\dots,M} |c_k|$ (including the unit circle)

$H_i(z)$ is both causal and stable

Case 2 : ROC of $H_i(z)$: $|z| < \min_{k=1,2,\dots,M} |c_k|$

$H_i(z)$ is both non-stable and non-stable

On the other hand, if there is an inverse system that is both stable and causal, then all the zeros c_k of H(z) must be inside the unit circle.

Reason:

$H_i(z)$ has a ROC: $|z| > \max_{k=1,2,\cdots,M} |c_k|$, and $\max_k |c_k| < 1$ due to $H_i(z)$ being stable.

In summary, a minimum-phase system:
- stable
- causal
- All its zeros and poles are within the unit circle

  - Its inverse system has the same properties as above (i.e., both causal and stable, with all its zeros and poles being inside the unit circle)

- Frequency Response of Rational System Functions

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - d_k e^{-j\omega})}$$

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \cdot \frac{\prod_{k=1}^{M} |1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N} |1 - d_k e^{-j\omega}|}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot H^*(e^{j\omega})$$

If $z = x + jy$, $z^* = x - jy$, $z \cdot z^* = x^2 - (jy)^2 = x^2 + y^2 = |z|^2$

$$|H(e^{j\omega})|^2 = \left|\frac{b_0}{a_0}\right|^2 \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})(1 - c_k^* e^{j\omega})}{\prod_{i=1}^{M} (1 - d_k e^{-j\omega})(1 - d_k^* e^{j\omega})}$$

Phase Response:

$$H(e^{jw}) = \frac{\sum_{k=0}^{M} b_k e^{-jwk}}{\sum_{k=0}^{N} a_k e^{-jwk}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k e^{-jw})}{\prod_{k=1}^{N} (1 - d_k e^{-jw})}$$

$$\arg[H(e^{jw})] = \arg\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \arg[1 - c_k e^{-jw}] - \sum_{k=1}^{N} \arg[1 - d_k e^{-jw}]$$

Since $z_1 = r_1 e^{j\theta_1}$, $z_2 = r_2 e^{j\theta_2}$

$$\arg[z_1 \cdot z_2] = \arg[r_1 r_2 e^{j(\theta_1 + \theta_2)}] = \theta_1 + \theta_2 = \arg[z_1] + \arg[z_2]$$

$$\arg\left[\frac{z_1}{z_2}\right] = \arg\left[\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right] = \arg\left[\frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}\right] = \theta_1 - \theta_2 = \arg[z_1] - \arg[z_2]$$

- All-Pass System (constant magnitude gain regardless of the frequency ω)

$$H(z) = \frac{z - a}{z - \frac{1}{a}}$$, where a is a real number

$$H(e^{jw}) = \frac{e^{jw} - a}{e^{jw} - \frac{1}{a}}$$

e.g., $a = 2$, $H(e^{jw}) = \dfrac{e^{jw} - 2}{e^{jw} - 0.5}$

$$|H(e^{jw})| = ?$$

$$|H(e^{j\omega})| = \frac{|e^{j\omega} - 2|}{|e^{j\omega} - 0.5|} = \frac{|\cos\omega + j\sin\omega - 2|}{|\cos\omega + j\sin\omega - 0.5|}$$

$$= \frac{\sqrt{(\cos\omega - 2)^2 + \sin^2\omega}}{\sqrt{(\cos\omega - 0.5)^2 + \sin^2\omega}}$$

$$= \frac{\sqrt{1 - 4\cos\omega + 4}}{\sqrt{1 - \cos\omega + 0.25}} = \frac{\sqrt{5 - 4\cos\omega}}{\sqrt{1.25 - \cos\omega}}$$

$$= \sqrt{4}\,\frac{\sqrt{1.25 - \cos\omega}}{\sqrt{1.25 - \cos\omega}}$$

$$|H(e^{j\omega})| = 2$$

- freqz function in Matlab

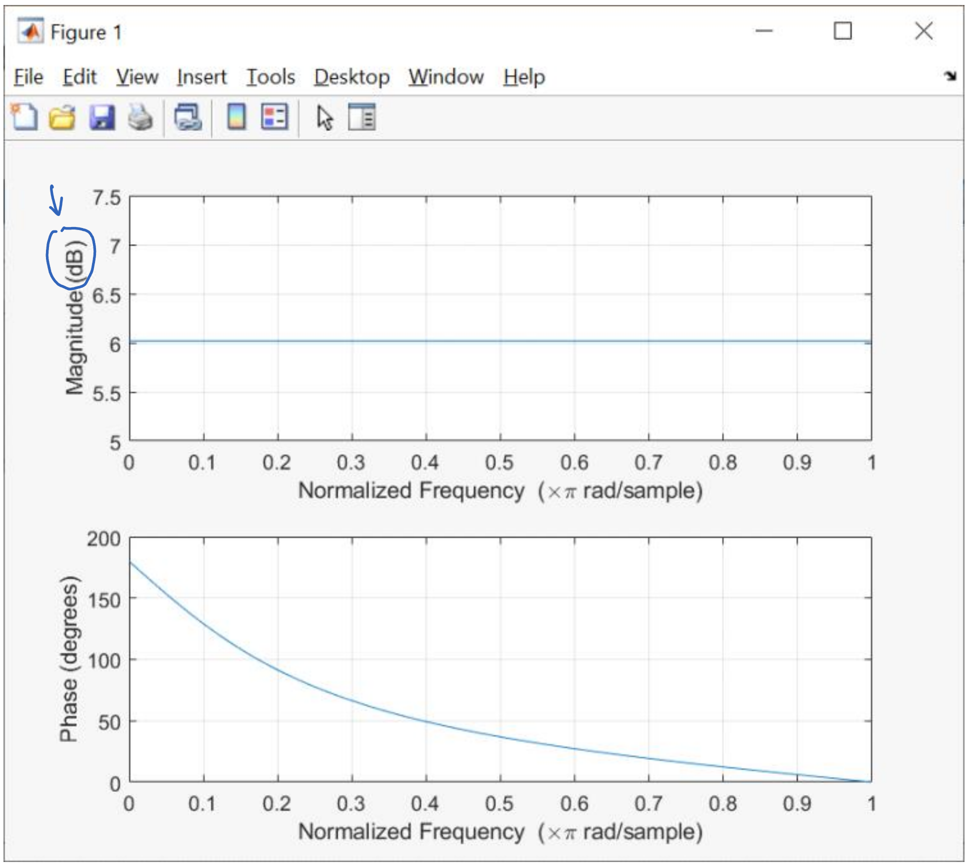
$$H(z) = \frac{z - a}{z - \frac{1}{a}}, \quad \text{if } a = 2, \quad H(z) = \frac{z-2}{z-0.5}$$

$$H(z) = \frac{z^{-1}(1 - 2z^{-1})}{z^{-1}(1 - 0.5z^{-1})} = \frac{1 - 2z^{-1}}{1 - 0.5z^{-1}}$$

In general, $$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Thus $$\begin{cases} b_0 = 1, & b_1 = -2, & M = 1 \\ a_0 = 1, & a_1 = -0.5, & N = 1 \end{cases}$$

```
>> doc freqz
>> b = [1 -2];
>> a = [1 -0.5];
>> freqz(b,a)
```




```
>> 20*log10(2)
ans =
   6.0206
```
dB

Log Magnitude (dB) $\triangleq$ gain in dB

$$20 \cdot \log_{10} |H(e^{jw})|$$

$$= 20 \log_{10} \left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20 \log_{10} |1 - c_k e^{-jw}| - \sum_{k=1}^{N} 20 \log_{10} |1 - d_k e^{-jw}|$$