Lecture 16

- All-Pass Systems

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}}, \quad a^* \text{ is the conjugate of } a.$$

$$= \frac{z}{z} \cdot \frac{z^{-1} - a^*}{1 - a z^{-1}} = \frac{1 - a^* z}{z - a} = \frac{-a^*\left(z - \frac{1}{a^*}\right)}{z - a}$$

$$\left.\begin{array}{ll} \text{zero:} & \frac{1}{a^*} \\ \text{pole:} & a \end{array}\right\} \Rightarrow \begin{array}{l} \left(\frac{1}{\text{zero}}\right)^* \rightarrow \text{pole} \\ \left(\frac{1}{\text{pole}}\right)^* \rightarrow \text{zero} \end{array}$$

$$z = e^{j\omega}$$

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - a e^{-j\omega}} = \frac{e^{-j\omega}(1 - a^* e^{j\omega})}{1 - a e^{-j\omega}}$$

$$|H_{ap}(e^{j\omega})| = \frac{|e^{-j\omega}| \cdot \overbrace{|1 - a^* e^{j\omega}|}^{A}}{\underbrace{|1 - a e^{-j\omega}|}_{A^*}} = 1$$

Why?

$$|e^{-j\omega}| = 1$$

Let $A = 1 - a^* e^{j\omega}$, then $A^* = 1 - (a^* e^{j\omega})^* = 1 - a e^{-j\omega}$

Thus $\frac{|A|}{|A^*|} = 1$ $\qquad |x + jy| = |x - jy|$ in general

- General Form of All-Pass Systems

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{z^{-1} - e_k^*}{1 - e_k z^{-1}} \cdot \frac{z^{-1} - e_k}{1 - e_k^* z^{-1}}$$

d_k's : real poles $\qquad$ e_k's : complex poles

e_k^*'s : complex poles

Example : $M_r = 2$, $d_1 = \frac{1}{2}$, $d_2 = -\frac{3}{4}$

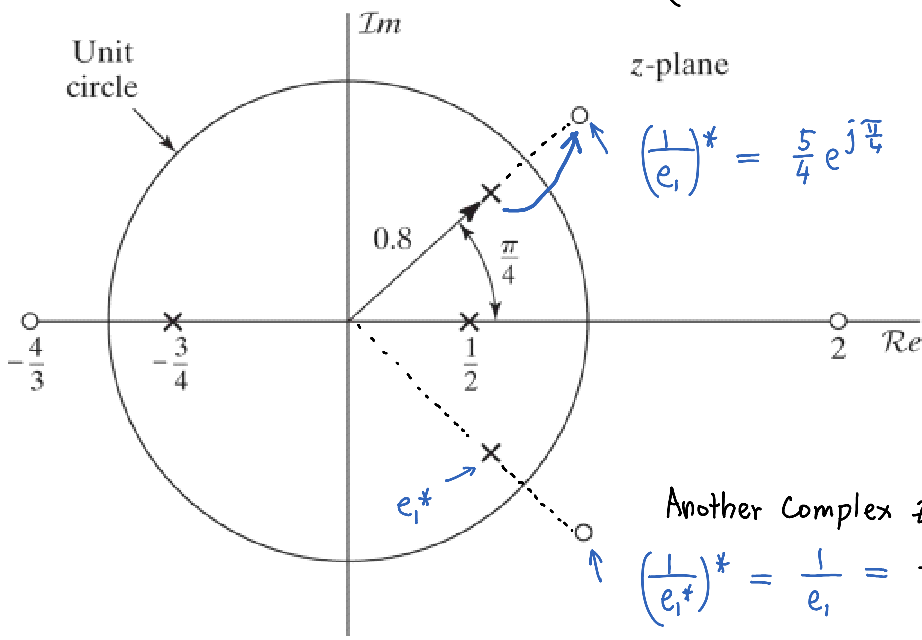
Two real poles

Two real zeros : $\begin{cases} \frac{1}{\frac{1}{2}} = 2 \\ \frac{1}{-\frac{3}{4}} = -\frac{4}{3} \end{cases}$

$M_c = 1$, $e_1 = 0.8\, e^{j\frac{\pi}{4}}$, $e_1^* = 0.8\, e^{-j\frac{\pi}{4}}$

Two complex zeros : $\left(\frac{1}{e_1}\right)^* = \left(\frac{1}{0.8\, e^{j\frac{\pi}{4}}}\right)^* = \left(\frac{5}{4} e^{-j\frac{\pi}{4}}\right)^* = \frac{5}{4} e^{j\frac{\pi}{4}}$

Figure 5.18 Typical pole–zero plot for an all-pass system.

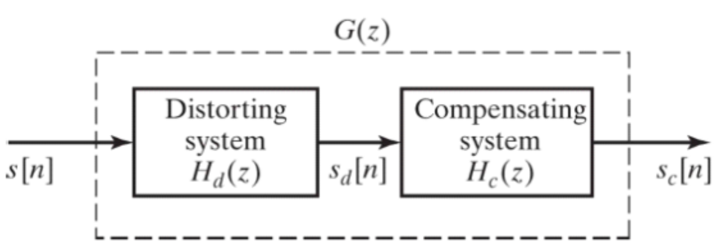


$\left(\frac{1}{e_1}\right)^* = \frac{5}{4} e^{j\frac{\pi}{4}}$

Another Complex zero:

$\left(\frac{1}{e_1^*}\right)^* = \frac{1}{e_1} = \frac{1}{0.8\, e^{j\frac{\pi}{4}}} = \frac{5}{4} e^{-j\frac{\pi}{4}}$

Application: Distortion Compensation by Linear Filtering



$G(z) = H_{ap}(z) = 1$

$H_d(z)$: non-minimum-phase system

One option : $G(z) = H_d(z) \cdot H_c(z) = 1 \Rightarrow H_c(z) = \frac{1}{H_d(z)}$

Inverse System of $H_d(z)$

Second Option : What if $\underset{\uparrow}{H_d(z)} = \underbrace{H_{d_{min}}(z)} \cdot \underbrace{H_{ap}(z)}$?

not minimum-phase — minimum-phase — all-pass system

Then we can choose the $\underbrace{\text{corresponding}}_{\text{compensating}}$ filter to be

$$H_c(z) = \frac{1}{H_{d_{min}}(z)}$$

Such that

$$G(z) = H_d(z) \cdot H_c(z) = H_{d_{min}}(z) \cdot H_{ap}(z) \cdot \frac{1}{H_{d_{min}}(z)} = H_{ap}(z)$$

Example: Decomposition of a non-minimum-phase system as a minimum-phase system, followed by an all-pass system.

$$H(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}} = \frac{z + 3}{z + \frac{1}{2}}$$

zero : -3 (outside the unit circle)
pole : $-\frac{1}{2}$

↑ non-minimum-phase

Goal : $H(z) = H_{min}(z) \cdot H_{ap}(z)$

In general , $H(z)$ has a zero c^* outside the unit circle

↓

Move the zero c^* to the all-pass system $H_{ap}(z)$ as a new zero

↓

Pair up the newly introduced zero c^* with a $\underbrace{\text{new pole}}$: $\left(\frac{1}{c^*}\right)^* = \frac{1}{c}$.

↓

Compensate for this new pole with a zero at $\frac{1}{c}$ ← inside the unit circle

and put this $\underbrace{\text{zero}}_{\text{inside the unit circle}}$ into $H_{min}(z)$

$$H(z) = \frac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}}$$

$1+3z^{-1}$: zero at (-3) : outside the unit circle

$1+\frac{1}{2}z^{-1}$: pole at $(-\frac{1}{2})$ inside the unit circle

$$= H_{min}(z) \cdot H_{ap}(z)$$

$$= \underbrace{\frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}}}_{\text{minimum-phase}} \; \underbrace{\frac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}}}_{\text{All-pass}}$$

new pole at $(-\frac{1}{3})$

Compared with

- All-Pass Systems

$$H_{ap}(z) = \frac{z^{-1}-a^*}{1-az^{-1}}$$

$$|H_{ap}(z)| = 1$$

$H_{ap}(z)$ in the above example :

$$\frac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}} = \frac{3(z^{-1}+\frac{1}{3})}{1-(-\frac{1}{3})z^{-1}}$$

$a = a^* = -\frac{1}{3}$

$$\left|\frac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}}\right| = 3 \cdot \underbrace{\left|\frac{z^{-1}-(\overset{\sim}{-\frac{1}{3}})}{1-(-\frac{1}{3})z^{-1}}\right|}_{1}$$

$$= 3$$

All-pass system with

$$|H_{ap}(z)| = 3$$

Thus

$$H(z) = 3 \cdot \underbrace{\frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}}}_{H_{min}(z)} \cdot \underbrace{\frac{z^{-1}+\frac{1}{3}}{1+\frac{1}{3}z^{-1}}}_{H_{ap}(z)}$$

and $|H_{ap}(z)| = 1$

Another Example:

$$H(z) = \frac{(1+\frac{3}{2}e^{j\frac{\pi}{4}}z^{-1})(1+\frac{3}{2}e^{-j\frac{\pi}{4}}z^{-1})}{1-\frac{1}{3}z^{-1}} = H_{min}(z)\cdot H_{ap}(z)$$

Two zeros : $-\frac{3}{2}e^{j\frac{\pi}{4}}$, $-\frac{3}{2}e^{-j\frac{\pi}{4}}$ both are outside the unit circle.

One pole : $\frac{1}{3}$ (inside the unit circle)

$$H(z) = \frac{?}{1-\frac{1}{3}z^{-1}} \cdot \underbrace{\frac{?}{?}}_{H_{ap}(z)}$$

Align with the form : $H_{ap}(z) = \dfrac{z^{-1}-a^*}{1-az^{-1}}$

$$1+\frac{3}{2}e^{j\frac{\pi}{4}}z^{-1} = \frac{3}{2}e^{j\frac{\pi}{4}}\left(z^{-1}+\frac{1}{\frac{3}{2}e^{j\frac{\pi}{4}}}\right) = \underbrace{\frac{3}{2}e^{j\frac{\pi}{4}}}_{\text{moved to } H_{min}(z)}\left[z^{-1}-\left(-\frac{2}{3}e^{-j\frac{\pi}{4}}\right)\right]$$

$$1+\frac{3}{2}e^{-j\frac{\pi}{4}}z^{-1} = \frac{3}{2}e^{-j\frac{\pi}{4}}\left(z^{-1}+\frac{1}{\frac{3}{2}e^{-j\frac{\pi}{4}}}\right) = \underbrace{\frac{3}{2}e^{-j\frac{\pi}{4}}}_{\text{moved to } H_{min}(z)}\left[z^{-1}-\left(-\frac{2}{3}e^{j\frac{\pi}{4}}\right)\right]$$

Thus

$$H_{ap}(z) = \frac{\left[z^{-1}-\left(-\frac{2}{3}\cdot e^{-j\frac{\pi}{4}}\right)\right]\cdot\left[z^{-1}-\left(-\frac{2}{3}\cdot e^{j\frac{\pi}{4}}\right)\right]}{\left[1-\underbrace{\left(-\frac{2}{3}e^{j\frac{\pi}{4}}\right)}_{\left(-\frac{2}{3}e^{-j\frac{\pi}{4}}\right)^*}z^{-1}\right]\cdot\left[1-\underbrace{\left(-\frac{2}{3}e^{-j\frac{\pi}{4}}\right)}_{\left(-\frac{2}{3}e^{j\frac{\pi}{4}}\right)^*}z^{-1}\right]}$$

$$H_{ap}(z) = \frac{\left(z^{-1}+\frac{2}{3}e^{-j\frac{\pi}{4}}\right)\left(z^{-1}+\frac{2}{3}e^{j\frac{\pi}{4}}\right)}{\left(1+\frac{2}{3}e^{j\frac{\pi}{4}}z^{-1}\right)\left(1+\frac{2}{3}e^{-j\frac{\pi}{4}}z^{-1}\right)}$$

And

$$H_{min}(z) = \frac{\left(1+\frac{2}{3}e^{j\frac{\pi}{4}}z^{-1}\right)\left(1+\frac{2}{3}e^{-j\frac{\pi}{4}}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}\cdot\underbrace{\frac{3}{2}e^{j\frac{\pi}{4}}\cdot\frac{3}{2}e^{-j\frac{\pi}{4}}}_{\frac{9}{4}}$$

In summary,

$$H(z) = \frac{\left(1+\frac{3}{2}e^{j\frac{\pi}{4}}z^{-1}\right)\left(1+\frac{3}{2}e^{-j\frac{\pi}{4}}z^{-1}\right)}{1-\frac{1}{3}z^{-1}} = H_{min}(z)\cdot H_{ap}(z)$$

where

$$H_{min}(z) = \frac{9}{4}\,\frac{\left(1+\frac{2}{3}e^{j\frac{\pi}{4}}z^{-1}\right)\left(1+\frac{2}{3}e^{-j\frac{\pi}{4}}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}$$

$$H_{ap}(z) = \frac{\left(z^{-1}+\frac{2}{3}e^{-j\frac{\pi}{4}}\right)\left(z^{-1}+\frac{2}{3}e^{j\frac{\pi}{4}}\right)}{\left(1+\frac{2}{3}e^{j\frac{\pi}{4}}z^{-1}\right)\left(1+\frac{2}{3}e^{-j\frac{\pi}{4}}z^{-1}\right)}$$