Lecture 2

- Linear / Non-linear Systems

$x_1[n] \rightarrow \boxed{T\{\cdot\}} \rightarrow y_1[n]$ $\qquad$ $x_2[n] \rightarrow \boxed{T\{\cdot\}} \rightarrow y_2[n]$

$$a x_1[n] + b x_2[n] \rightarrow \boxed{T\{\cdot\}} \xrightarrow{?} a y_1[n] + b y_2[n]$$

If $T\{a x_1[n] + b x_2[n]\} = a T\{x_1[n]\} + b T\{x_2[n]\}$

for all a and b, then the system is linear.

Example : $y[n] = \sum_{k=-\infty}^{n} x[k]$ $\quad$ Accumulator System

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k], \quad y_2[n] = \sum_{k=-\infty}^{n} x_2[k]$$

$$x_3[n] = a x_1[n] + b x_2[n]$$

$$y_3[n] = \sum_{k=-\infty}^{n} x_3[k] = \sum_{k=-\infty}^{n} \{a x_1[k] + b x_2[k]\}$$

$$= a \sum x_1[k] + b \sum x_2[k]$$

$$= a y_1[n] + b y_2[n]$$

The system is linear.

- Time-Invariant (TI) Systems

$$x[n] \rightarrow \boxed{T\{\cdot\}} \rightarrow y[n]$$

$x_1[n] = x[n - n_0] \rightarrow$ $\qquad$ $\rightarrow y_1[n] \overset{?}{=} y[n - n_0]$

$\uparrow$ arbitrary integer $\neq 0$

If "=" then Shift-invariant or T-I

Example : $y[n] = \sum_{k=-\infty}^{n} x[k]$

$$x_1[n] = x[n-n_0], \quad y_1[n] = \sum_{k=-\infty}^{\infty} x_1[k] = \sum_{k=-\infty}^{\infty} x[\underbrace{k-n_0}_{k_1}]$$

Let $k_1 = k - n_0$

Since $-\infty \leq k \leq n$, $\quad -\infty \leq k_1 = k - n_0 \leq n - n_0$

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] = y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

Since $y[n] = \sum_{k=-\infty}^{n} x[k]$

Yes, the system is T-I.

Another example: $\quad y[n] = x[2n]$

$$x_1[n] = x[n-n_0], \quad y_1[n] = x_1[2n] = x[2n-n_0] = x[2(n-\tfrac{n_0}{2})]$$

Check if $y_1[n] \overset{?}{=} y[n-n_0] = x[2(n-n_0)]$

$\neq$

Hence, the system is Time-Variant (NOT T-I).

- Stability (BIBO)

BI: Bounded Input, BO: Bounded Output

If $|x[n]| \leq B_x$, and $|y[n]| \leq B_y$, both B_x and $B_y < \infty$

then the system is stable.

Example: $\quad y[n] = \{x[n]\}^2 \quad \forall n$ (for all n values)

If $|x[n]| \leq B_x$, $\quad |y[n]| = |(x[n])^2| = \{|x[n]|\}^2 \leq B_x^2 = B_y$

Yes, stable.

Another example: $\quad y[n] = \sum_{k=-\infty}^{n} x[k] \quad$ non-stable.

Pick an input $x[n] = u[n]$, $\quad |u[n]| \leq 1$, $\quad y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0, & n<0 \\ n+1, & n \geq 0 \end{cases}$

$n \to \infty$, $\quad y[n] = n+1 \to \infty$

- Causality
  If the output of the system depends on the current or previous inputs only, then the system is causal.

Example : $\underset{\text{current}}{y[n]} = \underset{\text{current}}{x[n]} - \underset{\text{previous input}}{x[n-1]}$ Causal

$y[n] = \underset{\text{future}}{x[n+1]} - x[n]$ non - causal

- Convolution Sum

- Linear Time Invariant (LTI) Systems $h[n] = T\{\delta[n]\}$

$$y[n] = \underset{\text{convolution sum}}{x[n] * h[n]} = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$= h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

Interpretation of Convolution Sums

- Superposition

$$y[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{x[k]\text{ : const}} \cdot \underbrace{h[n-k]}_{\text{sequence (a shifted version of } h[n])}$$

$$= x[0] \cdot h[n] + x[1] \cdot h[n-1] + x[2] \cdot h[n-2] + \cdots$$
$$+ x[-1] \cdot h[n+1] + x[-2] \cdot h[n+2] + \cdots$$

- Computational (flipping, shifting)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \underbrace{h[n-k]}_{h[\underset{\text{flipping}}{-}\overset{\text{shifting}}{(k-n)}]}$$

Flipping example : $z[n] = \{ \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots \}$

$$\begin{array}{cccccccc} n \Rightarrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array}$$

$$z[-n] = \{ \cdots, 3, 2, 1, 0, -1, -2, -3, \cdots \}$$

$$z[n-1] = \{ \cdots, -3, -2, -1, 0, 1, 2, \cdots \}$$

$\uparrow$ Shifted version of $z[n]$
$\underbrace{\text{Shifted}}_{\text{delayed by one time point}}$

Example 2.11 (pp 27)

$$h[n] = u[n] - u[n-N] = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere} \end{cases}$$

e.g., $N = 2$

$$u[n] - u[n-2] = \begin{cases} 1, & 0 \leq n \leq 2-1 = 1 \\ 0, & \text{elsewhere} \end{cases}$$

$\cdots$	-3	-2	-1	0	1	2	$\cdots$		n
$\cdots$	$0,$	$0,$	$0,$	$1,$	$1,$	$1,$	$\cdots$		$u[n]$
$\cdots$	$0,$	$0,$	$0,$	$0,$	$0,$	$1,$	$1,$	$\cdots$	$u[n-2]$
	$0,$	$0,$	$0,$	$1,$	$1,$	0	0	$\cdots$	$u[n]-u[n-2]$

Input $x[n] = a^n u[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

$$\begin{array}{lccccccc} & \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \quad n \\ \text{e.g., } a = 0.5, \quad x[n] = & \cdots & 0, & 0, & 0, & 1, & 0.5, & 0.25, \cdots \end{array}$$

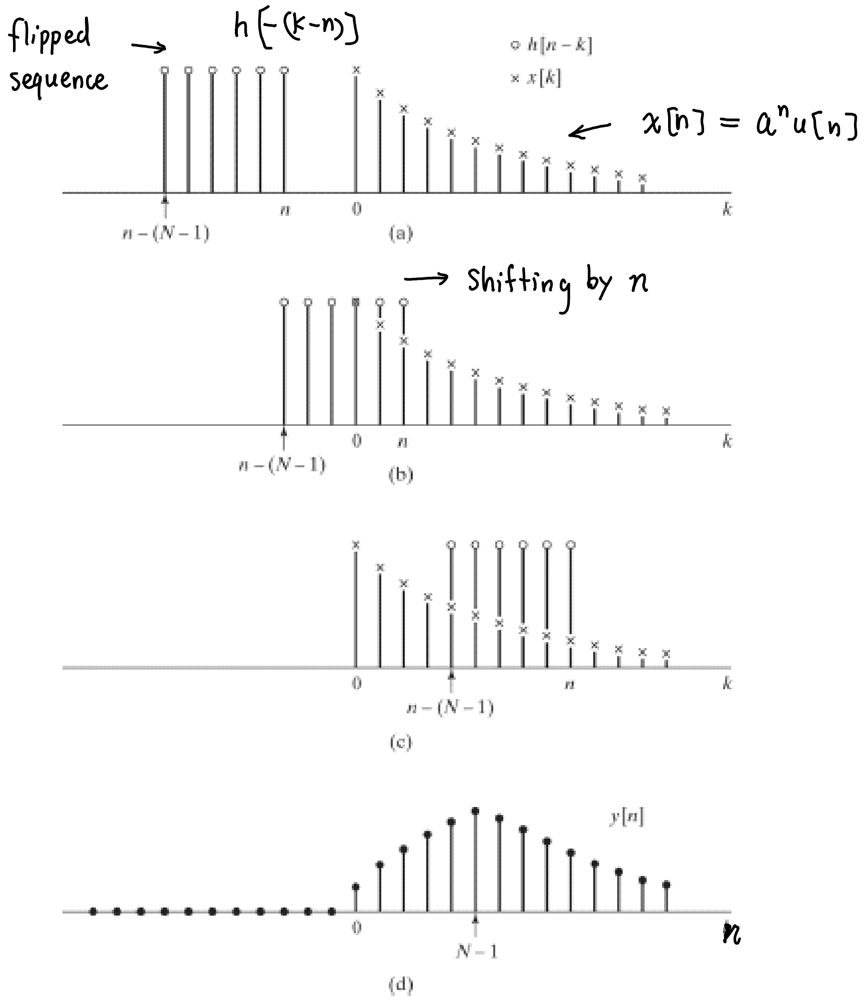
Derivation : $y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \underbrace{h[n-k]}_{\text{focus on } h[n-k] \neq 0}$

For $h[n-k] = 1$, $\quad 0 \leq n-k \leq N-1$

$$\Rightarrow \begin{cases} k \leq n \\ k \geq n-(N-1) \Rightarrow k \geq n-N+1 \end{cases}$$

$$\Rightarrow n-N+1 \leq k \leq n$$

How about $x[k]$? $\quad k \geq 0$, $x[k] \neq 0$; otherwise $x[k] = 0$



Derivation (algebraic) (cont'd) :

$$\begin{cases} 0 \le k \\ n-N+1 \le k \le n \end{cases} \Rightarrow \max(0,\ n-N+1) \le k \le n$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]$$

Cases :

(1) $n<0$, $\quad \overbrace{\max(0,\ n-N+1)}^{0} \le k \le n < 0$

$x[k]\cdot h[n-k] = 0$, $\quad y[n] = 0$

(2) $0 < n \le N-1$, $\quad n-(N-1) \le 0$, thus $\max(0,\ n-(N-1)) = 0$

Thus $0 \le k \le n \le N-1$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$= \sum_{k=0}^{n} x[k] \cdot 1 = \sum_{k=0}^{n} a^k = 1 + a + a^2 + \cdots + a^n$$

geometric series $= \frac{1(1-a^{n+1})}{1-a} = \frac{1-a^{n+1}}{1-a}$.

(3) $n > N-1 \Rightarrow n-(N-1) > 0 \Rightarrow \max(0, n-(N-1)) = n-(N-1)$

Thus $n-(N-1) \leq k \leq n$

$$y[n] = \sum_{k=n-(N-1)}^{n} a^k \cdot 1 = \frac{a^{n-N+1} - a^{n+1}}{1-a}.$$

In general, $\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a}$, $N_2 \geq N_1$

In Summary

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1-a^{n+1}}{1-a}, & 0 \leq n \leq N-1 \\ a^{n-N+1}\left(\frac{1-a^N}{1-a}\right), & n > N-1 \end{cases}$$

- Matlab function conv( )

```
>> n = 0: 20;
>> x = (0.5).^n;
>> h = [1, 1, 1, 1];
>> y = conv(x, h);
>> stem(y); grid
```