Lecture 3

Tests for Stability and Causality

Stable : $B_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty \Rightarrow$ Stable System

Causal : $h[n] = 0, \quad n < 0$

Examples :

(1) $y[n] = x[n-1]$: Ideal Delay System

Q: $h[n]$?

$$h[n] = T\{x[n] = \delta[n]\} = \delta[n-1] = \begin{cases} 1, & n = 1 \\ 0, & \text{elsewhere} \end{cases}$$

Both stable and causal.

(2) $$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x(n-k), \quad M_1 > 0,\ M_2 > 0, \text{ integers}$$

e.g., $M_1 = M_2 = 1$, $y[n] = \frac{1}{3} \sum_{k=-1}^{1} x[n-k]$

$$= \frac{1}{3}\{x[n-1] + x[n] + x[n+1]\}$$

Q: $h[n]$?

$$h[n] = \frac{1}{3} \sum_{k=-1}^{1} \delta[n-k] = \begin{cases} \frac{1}{3}, & n = -1, 0, 1 \\ 0, & \text{elsewhere} \end{cases}$$

Stable, non-causal system

- Fourier Transforms (DTFT) : $x[n] \longleftrightarrow X(e^{j\omega})$

Discrete Time

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n} \quad : \text{Forward Transform}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} d\omega \quad : \text{Inverse Transform}$$

$X(e^{j\omega})$ is periodic with fundamental period being 2π.

$$X(e^{j(\omega+k\cdot 2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j(\omega+2k\pi)n}$$

$$= \sum_{n} x[n]\cdot e^{-j\omega n}\cdot \underbrace{e^{-j2k\pi n}}_{1}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\omega n}$$

$$= X(e^{j\omega})$$

In general, $X(e^{j\omega}) = X_R(e^{j\omega}) + j X_I(e^{j\omega})$

$$X(e^{j\omega}) = |X(e^{j\omega})|\, e^{j\measuredangle X(e^{j\omega})}$$

Example: $x[n] = a^n u[n]$, $|a|<1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\omega n} = \sum_{n=0}^{\infty} a^n\cdot e^{-j\omega n} = \sum_{n=0}^{\infty}\left(\frac{a}{e^{j\omega}}\right)^n$$

$$= 1 + \frac{a}{e^{j\omega}} + \left(\frac{a}{e^{j\omega}}\right)^2 + \cdots$$

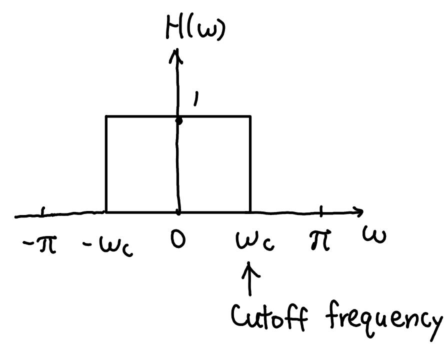
$$= \frac{1}{1-ae^{-j\omega}} \qquad \text{where } \left|\frac{a}{e^{j\omega}}\right| = \frac{|a|}{|e^{j\omega}|} = |a| < 1$$

In general, $\sum_{n=0}^{\infty} \beta^n = 1+\beta+\beta^2+\cdots+\ldots = \frac{1}{1-\beta}$, $|\beta|<1$

Example: Inverse DTFT

$$H(\omega) = \begin{cases} 1, & |\omega| \leqslant \omega_c < \pi \\ 0, & \text{elsewhere} \end{cases}$$

$h[n]$?



$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\cdot e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

Table 2.3 FOURIER TRANSFORM PAIRS

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Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $\boxed{a^n u[n] \quad (\lvert a\rvert < 1)}$	$\boxed{\dfrac{1}{1-ae^{-j\omega}}}$
5. $u[n]$	$\dfrac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{(1-ae^{-j\omega})^2}$
7. $\dfrac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\dfrac{1}{1-2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
→ 8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

(Cont'd) Inverse DTFT:

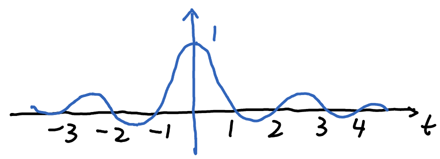
$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(w)\cdot e^{jwn}\,dw = \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{jwn}\,dw$$

$$= \frac{1}{2\pi}\left(\frac{e^{jwn}}{jn}\right)\Bigg|_{-w_c}^{w_c} = \frac{1}{2\pi}\cdot\frac{e^{jw_c n} - e^{-jw_c n}}{jn}$$

Recall: $e^{j\theta} = \cos\theta + j\sin\theta$, $\quad e^{j\theta} - e^{-j\theta} = 2j\sin\theta$

$$h[n] = \frac{1}{2\pi}\cdot\frac{2j\sin(w_c n)}{jn} = \frac{\sin(w_c n)}{\pi n}$$

$$\text{Sinc}(t) = \frac{\sin(\pi \cdot t)}{\pi t}$$



$$\lim_{t\to 0} \text{Sinc}(t) = \lim_{t\to 0} \frac{\sin(\pi t)}{\pi t} = \lim_{t\to 0} \frac{\pi \cos(\pi t)}{\pi} = \lim_{t\to 0} \cos(\pi t) = \cos 0 = 1$$

$$\text{Sinc}(t)\Big|_{t=1} = \frac{\sin(\pi \cdot 1)}{\pi \cdot 1} = 0, \quad \text{sinc}(2) = \frac{\sin(2\pi)}{2\pi} = 0$$

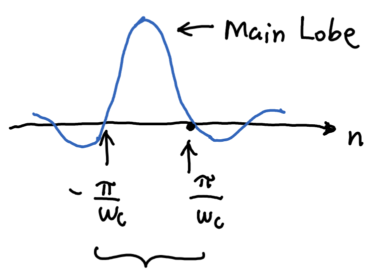
$$h[n] = \frac{\sin(\omega_c n)}{\pi n} = \frac{\sin\left(\pi \cdot \frac{\omega_c n}{\pi}\right)}{\frac{\pi}{\omega_c} \cdot (\omega_c n)} = \frac{1}{\frac{\pi}{\omega_c}} \cdot \text{Sinc}\left(\frac{\omega_c n}{\pi}\right)$$

$$= \frac{\omega_c}{\pi} \text{Sinc}\left(\frac{\omega_c n}{\pi}\right)$$

zero-crossing points: $h[n] = 0$

$$\frac{\omega_c n}{\pi} = 1 \Rightarrow n = \frac{\pi}{\omega_c}$$

$$\frac{\omega_c n}{\pi} = -1 \Rightarrow n = -\frac{\pi}{\omega_c}$$



$$\text{width} = \frac{2\pi}{\omega_c}, \quad \omega_c \downarrow \Rightarrow \text{width} \uparrow$$

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

$(x+jy)^* = x - jy$

complex sequence

Any sequence $x[n] = x_e[n] + x_o[n]$, where

$$x_e[n] = \frac{1}{2}\{x[n] + x^*[-n]\} \quad \cdots (1)$$

$$x_o[n] = \frac{1}{2}\{x[n] - x^*[-n]\} \quad \cdots (2)$$

(1) + (2) : $x_e[n] + x_o[n] = x[n]$

Properties : $x_e[n] = x_e^*[-n]$: conjugate - symmetric sequence

Proof : From (1) :

$$x_e[-n] = \frac{1}{2}\{x[-n] + x^*[-(-n)]\}$$

$$= \frac{1}{2}\{x[-n] + x^*[n]\}$$

$$x_e^*[-n] = \frac{1}{2}\{x^*[-n] + \{x^*[n]\}^*\}$$

$$= \frac{1}{2}\{x^*[-n] + x[n]\} = x_e[n]$$

If $x[n]$ is a real sequence, then $x^*[n] = x[n]$.

$x[n] = x_e[n] + x_o[n]$, where

$$\begin{cases} \underset{\uparrow \text{ even sequence}}{x_e[n]} = \frac{1}{2}\{x[n] + x[-n]\} \\ \underset{\uparrow \text{ odd sequence}}{x_o[n]} = \frac{1}{2}\{x[n] - x[-n]\} \end{cases}$$

We can show that

$$\begin{cases} x_e[n] = x_e[-n] \\ x_o[n] = -x_o[-n]. \end{cases}$$