Lecture 4

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Property 1: $x^*[n] \xrightarrow{FT} X^*(e^{-jw})$

Proof: $x[n] \xrightarrow{FT} X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jwn}$

$$\downarrow$$

$$X(e^{-jw}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j(-w)n} = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{jwn}$$

$$\downarrow$$

$$X^*(e^{-jw}) = \left(\sum_{n=-\infty}^{\infty} x[n].e^{jwn}\right)^*$$

$$= \sum_{n=-\infty}^{\infty} \left(x[n].e^{jwn}\right)^*$$

$$= \sum_{n=-\infty}^{\infty} x^*[n] \cdot \left(e^{jwn}\right)^*$$

$$= \sum_{n=-\infty}^{\infty} x^*[n]\, e^{-jwn} = \mathcal{F}\{x^*[n]\}$$

where $y = \rho e^{j\theta} = \rho\cos\theta + \rho j \sin\theta$

$y^* = \rho\cos\theta - \rho j\sin\theta = \rho e^{-j\theta}$

Table 2.2 FOURIER TRANSFORM THEOREMS

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$\boxed{e^{-j\omega n_d} X(e^{j\omega})}$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j \dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n] y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) Y^*(e^{j\omega}) d\omega$$

Property 2 : $n_d = 1$

$$x[n] \xrightarrow{FT} X(e^{jw})$$

$$y[n] = x[n-1] \xrightarrow{FT} Y(e^{jw}) = e^{-jw} X(e^{jw}) \; ?$$

Proof :

$$Y(e^{jw}) = \sum_{n=-\infty}^{\infty} y[n] e^{-jwn} = \sum_{n=-\infty}^{\infty} \underbrace{x[n-1]}_{m = n-1} e^{-jwn}$$

Let $m = n-1$, then $n = m+1$

$$-\infty < n < \infty \Rightarrow -\infty < m = n-1 < \infty$$

$$Y(e^{jw}) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-jw(m+1)}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-jwm} \cdot e^{-jw} = e^{-jw} \overbrace{\left[\sum_{m=-\infty}^{\infty} x[m] \cdot e^{-jwm} \right]}^{X(e^{jw})}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

Conv. sum

$$X(e^{jw}) \times H(e^{jw}) = Y(e^{jw})$$

Frequency Response of the System:

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$$

$$\Big\downarrow FT^{-1}$$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{jw})\, e^{jwn}\, dw$$

Example:

$$y[n] + \frac{1}{2} y[n-1] + \cdots = x[n] - \frac{1}{2} x[n-1] + \cdots$$

$$Y(e^{jw}) + \frac{1}{2} Y(e^{jw}) e^{-jw} + \cdots = X(e^{jw}) - \frac{1}{2} X(e^{jw}) e^{-jw} + \cdots$$

$$Y(e^{jw}) \left[1 + \frac{1}{2} e^{-jw} + \cdots\right] = X(e^{jw}) \left[1 - \frac{1}{2} e^{-jw} + \cdots\right]$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1 - \frac{1}{2} e^{-jw} + \cdots}{1 + \frac{1}{2} e^{-jw} + \cdots}$$

- Eigenfunctions of LTI Systems

$$x[n] = e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = ?$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k] \cdot h[k] = \sum_{k=-\infty}^{\infty} \underbrace{e^{j\omega(n-k)}}_{e^{j\omega n} \cdot e^{-j\omega k}} \cdot h[k]$$

$$= e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} \left\{ e^{-j\omega k} \cdot h[k] \right\}}$$

$$y[n] = \underbrace{e^{j\omega n}}_{x[n]} \cdot \underbrace{H(e^{j\omega})}_{\text{Frequency Response}}$$

$$y[n] = \underbrace{x[n]}_{\text{Eigenfunction}} \cdot \underbrace{H(e^{j\omega})}_{\text{Eigenvalue}}$$

Example : $x[n] = e^{j\frac{\pi}{2}n}$

$$y[n] = x[n] \cdot \underbrace{H(e^{j\frac{\pi}{2}})}_{\text{Constant}}$$

Another Example : $x[n] = \cos\left(\frac{\pi}{2}n\right) = \frac{1}{2}\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right)$

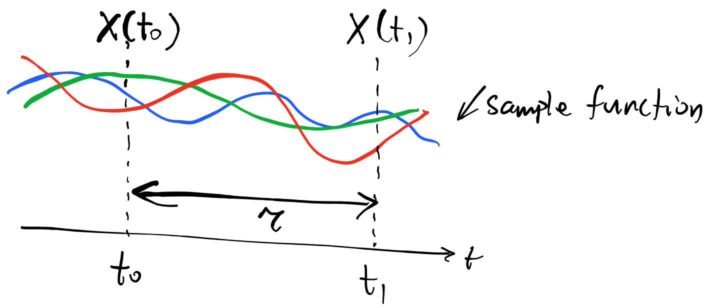
$$x_1[n] = e^{j\frac{\pi}{2}n} \longrightarrow y_1[n] = x_1[n]\, H(e^{j\frac{\pi}{2}})$$

$$x_2[n] = e^{-j\frac{\pi}{2}n} \longrightarrow y_2[n] = x_2[n]\, H(e^{-j\frac{\pi}{2}})$$

$$x[n] = \frac{1}{2}\left(x_1[n] + x_2[n]\right) \longrightarrow y[n] = \frac{1}{2}\left(y_1[n] + y_2[n]\right).$$

Similarly, $x[n] = \sin\left(\frac{\pi}{2}n\right) = \frac{1}{2j}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right).$

- Discrete-Time Random Signals
  Random Process X(t)



Random Variable: X

Discrete random variables
Continuous random variables : $X \sim N(\mu, \sigma)$ — μ: mean, σ: standard deviation

Autocorrelation Function $E[X(t_0) \cdot X(t_1)] = f(t_0, t_1)$

For Wide-Sense Stationary random processes:
(WSS)

$$\begin{cases} E[X(t_0) \cdot X(t_1)] = f(\underbrace{t_1 - t_0}_{\tau}) = f(\tau) \\ E[X(t_0)] = E[X(t_1)] = m_x(t) \text{ : constant, independent of } t. \end{cases}$$

- Autocorrelation of the output $y[n]$

$$E\{\underset{\substack{\uparrow \\ \text{random sequence}}}{y[n]} \cdot y[n+m]\} = \phi_{yy}[n, n+m]$$

Given the autocorrelation function of the input sequence, determine the autocorrelation of the output.

- Given the mean of the input, determine the mean of the output

Review some concepts in probability:
Two random variables: A and B

$E[A \cdot B] \neq E[A] \cdot E[B]$ in general; However, if A and B are independent
$E[A \cdot B] = E[A] \cdot E[B]$