# Lecture 5

- Given the mean of the input, determine the mean of the output.

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k] \cdot h[k]$$

$$\underbrace{E\{y[n]\}}_{m_y} = E\left\{ \sum_k x[n-k] \cdot \underbrace{h[k]}_{\text{deterministic}} \right\} = \sum_k h[k] \cdot \underbrace{E\{x[n-k]\}}_{\text{Given } m_x \text{ due to the W.S.S. assumption}}$$

$$m_y = \sum_{k=-\infty}^{\infty} m_x \cdot h[k]$$

$$= m_x \sum_{k=-\infty}^{\infty} h[k]$$

- Given the autocorrelation function of the input sequence, determine the autocorrelation function of the output

$$E\{y[n] \cdot y[n+m]\} \triangleq \phi_{yy}[n, n+m] \quad ?$$

$$\text{Given} \quad \phi_{xx}[n, n+m] = E\{x[n] \cdot x[n+m]\}.$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k], \quad y[n+m] = \sum_{r=-\infty}^{\infty} h[r] \cdot x[n+m-r]$$

$$\phi_{yy}[n, n+m] = E\left\{ \sum_k \sum_r h[k] \cdot h[r] \cdot x[n-k] \cdot x[n+m-r] \right\}$$

$$= \sum_k \sum_r h[k] \cdot h[r] \cdot \underbrace{E\{x[n-k]\, x[n+m-r]\}}_{\phi_{xx}[\underbrace{m+k-r}_{(n+m-r)-(n-k)}]}$$

$$\phi_{yy}[n, n+m] \triangleq \phi_{yy}[m]$$

$$= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\, \phi_{xx}[m+k-r]$$

$$\phi_{yy}[n, n+m] \triangleq \phi_{yy}[m]$$

$$= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[\underbrace{r}_{\ell+k}]\, \phi_{xx}[m + \underbrace{k-r}_{-\ell}]$$

Let $\ell = r - k$,

then $r = \ell + k$, and $-\ell = k - r$

$$\phi_{yy}[m] = \sum_{k=-\infty}^{\infty} h[k] \sum_{\ell=-\infty}^{\infty} h[\ell+k]\, \phi_{xx}[m-\ell]$$

$$= \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell] \underbrace{\sum_{k=-\infty}^{\infty} h[k] \cdot h[\ell+k]}_{c_{hh}[\ell]}$$

where $c_{hh}[\ell] = \sum_{k=-\infty}^{\infty} h[k] \cdot h[k+\ell]$

↑ deterministic autocorrelation sequence

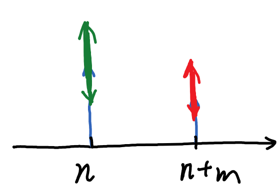
$$\phi_{yy}[m] = \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell] \cdot c_{hh}[\ell] = \phi_{xx}[m] * c_{hh}[m]$$

For comparison

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k] \cdot h[k] = x[n] * h[n]$$

- Power Density Spectrum

$$\phi_{xx}[m] \xrightarrow{FT} \underbrace{\Phi_{xx}(e^{j\omega})}_{PDS}$$




$\underbrace{\text{White Noise}}_{\text{random sequence}}$ $x[n]$, $\quad \phi_{xx}[m] = \sigma_x^2 \delta[m]$

$$\phi_{xx}[m] = E\{x[n] \cdot x[n+m]\} = \begin{cases} \sigma_x^2, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

If $m=0$, $\phi_{xx}[m] = \phi_{xx}[0] = E\{x[n]\cdot x[n+0]\} = \underbrace{E\{x^2[n]\}}_{\text{Mean Square}} = \sigma_x^2$

If $E\{x[n]\} = 0$, then $\text{var}\{x[n]\} \triangleq \sigma_x^2 = \phi_{xx}[0]$

In general, for any random variable $Z$

$\text{Var}[Z] = E[Z^2] - E^2[Z]$

If $E[Z] = 0$, then $\text{var}[Z] \triangleq \sigma_Z^2 = E[Z^2]$.

Go back to PSD

$$\phi_{yy}[m] = \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]\cdot c_{hh}[\ell] = \phi_{xx}[m] * c_{hh}[m]$$

$$\phi_{yy}[m] \xrightarrow{\mathcal{FT}} \Phi_{yy}(e^{j\omega}) \qquad \phi_{xx}[m] \xrightarrow{\mathcal{FT}} \Phi_{xx}(e^{j\omega}) \qquad c_{hh}[m] \xrightarrow{\mathcal{FT}} C_{hh}(e^{j\omega})$$

Thus $\Phi_{yy}(e^{j\omega}) = \Phi_{xx}(e^{j\omega})\cdot C_{hh}(e^{j\omega})$

where $C_{hh}(e^{j\omega}) = \mathcal{F}\{c_{hh}[m]\}$,

and $c_{hh}[m] = \sum_{k=-\infty}^{\infty} h[k]\cdot h[k+m]$

$= h[n] * h[-n]$

Since $h[n] * h[-n]$

$$= \sum_{k=-\infty}^{\infty} h[k]\cdot h[\underbrace{m-(-k)}_{m+k}]$$

To Summarize,

$$\begin{array}{ccccc} C_{hh}[m] & = & h[n] & * & h[-n] \\ \downarrow \mathcal{FT} & & \downarrow \mathcal{FT} & & \downarrow \mathcal{FT}\ ? \\ C_{hh}(e^{j\omega}) & = & H(e^{j\omega}) & \cdot & H^*(e^{j\omega}) \end{array}$$

$\uparrow$ Multiplication

By Property 2 of the symmetry properties of the Fourier Transforms:

$$x[n] \xrightarrow{\mathcal{FT}} X(e^{j\omega}), \quad \text{then} \quad x^*[-n] \xrightarrow{\mathcal{FT}} X^*(e^{j\omega})$$

If $h[n]$ is a real sequence, then $h[n] = h^*[n]$

Thus

$$h[n] \xrightarrow{\mathcal{FT}} H(e^{j\omega}), \quad \text{then} \quad h^*[-n] = h[-n] \xrightarrow{\mathcal{FT}} H^*(e^{j\omega})$$

Therefore,

$$C_{hh}(e^{j\omega}) = H(e^{j\omega}) \cdot H^*(e^{j\omega}) = |H(e^{j\omega})|^2$$

In general, if $z = x + jy$, $z^* = x - jy$

$$z \cdot z^* = x^2 + y^2 = |z|^2$$

$$\Phi_{yy}(e^{j\omega}) = \Phi_{xx}(e^{j\omega}) \cdot C_{hh}(e^{j\omega})$$

$$\boxed{\Phi_{yy}(e^{j\omega}) = \Phi_{xx}(e^{j\omega}) \cdot |H(e^{j\omega})|^2}$$

- Power Density Spectrum

$$x[n] \xrightarrow{\mathcal{FT}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

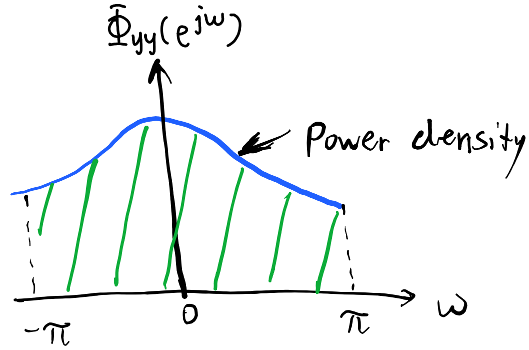
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

If $n = 0$, $\quad x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$

Look back at $\Phi_{yy}(e^{j\omega}) \xleftarrow{FT} \phi_{yy}[m] = E\{y[n] \cdot y[n+m]\}$

If $m = 0$, $\quad \phi_{yy}[0] = E\{y[n] \cdot y[n+0]\} = E\{y^2[n]\}$

$$\underbrace{E\{y^2[n]\}}_{\text{Average Power}} = \phi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{yy}(e^{j\omega})\, d\omega$$




For the white-noise sequence

$\phi_{xx}[m] = \sigma_x^2 \delta[m]$

$\downarrow FT$

$$\Phi_{xx}(e^{j\omega}) = FT\{\phi_{xx}[m]\} = \sum_{m=-\infty}^{\infty} \phi_{xx}[m] \cdot e^{-j\omega m} = \phi_{xx}[0] \cdot e^{-j\omega 0}$$

$$= \sigma_x^2 \quad \text{for all } \omega$$