# Lecture 6

Chapter 3: The Z-Transforms

DTFT: $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$ $\Rightarrow$ DTFT is the Z-T on the unit circle $|z| = 1$

$\downarrow$ Extension

Z-T: $X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$, where $z = \underbrace{re^{jw}}_{\text{polar form}} = r\cos w + jr\sin w$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot r^{-n} \cdot e^{-jwn}$$

$$= \sum_{n} \left(x[n] \cdot r^{-n}\right) \cdot e^{-jwn}$$

$$= \mathcal{F}_T\{\underbrace{x[n] \cdot r^{-n}}\}$$

: If $r = 1$, $X(z) = \mathcal{F}_T\{x[n] \cdot 1^{-n}\}$
$= \mathcal{F}_T\{x[n]\}$

For the Z-T to exist

$$\sum_{n=-\infty}^{\infty} \left|x[n] \cdot r^{-n}\right| < \infty \Rightarrow \sum_{n=-\infty}^{\infty} \left|x[n]\right| \cdot \underbrace{|z|^{-n}}_{r^{-n}} < \infty, \text{ Since } |z| = r$$

Find $\underbrace{\text{all the values of } z}_{\text{Regions of Convergence (ROC)}}$ such that the above condition is valid

Example 1: $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \frac{1}{1 - \frac{a}{z}}, \quad \text{if } \left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a|$$

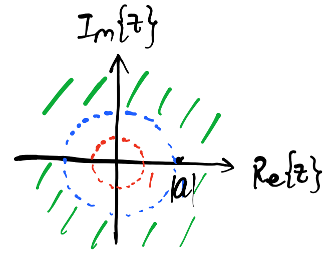
$$X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC}: |z| > |a|$$

$$X(z) = \frac{1}{1-az^{-1}} \xleftarrow{ZT} a^n u[n]$$

ROC: $|z| > |a|$

Several cases:

(1) $|a| > 1$, ROC: $|z| > |a| > 1$



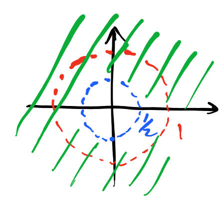

e.g. $x[n] = 2^n u[n] \longrightarrow$ DTFT does not exist

$\downarrow ZT$

$$X(z) = \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

ROC does not include the unit circle.

(2) $|a| < 1$, ROC: $|z| > |a| \Rightarrow$ DTFT exists




e.g., $x[n] = \left(\frac{1}{2}\right)^n u[n]$

$\downarrow$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC}: |z| > \frac{1}{2}.$$

$\downarrow z = e^{j\omega}$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

(3) $|a| = 1$, ROC: $|z| > |a| = 1$ outside the unit circle.

e.g., $x[n] = u[n] = 1^n u[n]$

$\uparrow a = 1$

$\downarrow$

$$X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$x[n]$ does not have DTFT

Next, determine the Z-T on $x[n] = -a^n u[-n-1]$ : Left-Sided Exponential Sequence

$$= \begin{cases} -a^n, & -n-1 \geq 0 \Rightarrow n \leq -1 \\ 0, & \text{elsewhere} \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{-1} -a^n \cdot z^{-n} = -\sum_{n=-\infty}^{-1} \left(a z^{-1}\right)^n$$

Let $m = -n$, $\quad -\infty < n \leq -1 \Rightarrow 1 \leq m = -n < \infty$

$$X(z) = -\sum_{m=1}^{\infty} \left(a z^{-1}\right)^{-m} = -\sum_{m=1}^{\infty} \Big(\underbrace{a^{-1} z}_{\beta}\Big)^m$$

**Formula**: $\displaystyle\sum_{m=0}^{\infty} \beta^m = \frac{1}{1-\beta}$, $\quad |\beta| < 1$

$$\sum_{m=1}^{\infty} \beta^m = \sum_{m=0}^{\infty} \beta^m - \sum_{m=0}^{0} \beta^m = \frac{1}{1-\beta} - 1$$

Thus $\displaystyle X(z) = -\left(\frac{1}{1-a^{-1}z} - 1\right)$, $\quad |a^{-1}z| < 1$

$$\Rightarrow \boxed{|z| < |a|}$$

$$= \frac{1}{1 - a z^{-1}}$$

since $$-\left(\frac{1}{1-a^{-1}z} - 1\right) = -\frac{1-(1-a^{-1}z)}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z}$$

$$= \frac{a^{-1}z}{a^{-1}z - 1} = \frac{a^{-1}z / a^{-1}z}{a^{-1}z/a^{-1}z - 1/a^{-1}z}$$

$$= \frac{1}{1 - a z^{-1}}$$

In summary, $$-a^n u[-n-1] \xrightarrow{ZT} \frac{1}{1-az^{-1}}, \quad |z| < |a|.$$

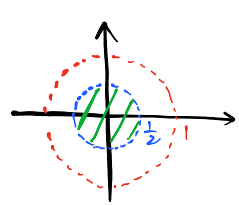
Similarly, there are several cases: $|a| < 1$, $|a| = 1$, $|a| > 1$.

e.g., $|a| < 1$

$$x[n] = -\left(\tfrac{1}{2}\right)^n u[-n-1]$$

$\downarrow$

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad \underbrace{|z| < \tfrac{1}{2}}_{\text{excluding the unit circle}}$$




DTFT does not exist.

Example : $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

$$\left(\frac{1}{2}\right)^n u[n] \longrightarrow \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$\left(-\frac{1}{3}\right)^n u[n] \longrightarrow \frac{1}{1-\left(-\frac{1}{3}\right)z^{-1}}, \quad |z| > \left|-\frac{1}{3}\right| = \frac{1}{3}$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}$$

ROC : $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3} \Rightarrow |z| > \frac{1}{2}$.

**Table 3.1** SOME COMMON $z$-TRANSFORM PAIRS

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| Sequence | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 6. $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 7. $na^n u[n]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 8. $-na^n u[-n-1]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 9. $\cos(\omega_0 n)u[n]$ | $\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 10. $\sin(\omega_0 n)u[n]$ | $\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 11. $r^n\cos(\omega_0 n)u[n]$ | $\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 12. $r^n\sin(\omega_0 n)u[n]$ | $\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$ | $\frac{1-a^N z^{-N}}{1-az^{-1}}$ | $\lvert z\rvert > 0$ |

$\delta[n] \xrightarrow{ZT} ?$

$$\sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1 \cdot z^{-0} = 1, \quad \text{ROC: All } z$$

- Zeros and Poles

$$X(z) = \frac{P(z)}{Q(z)} \quad \begin{cases} \text{zeros: all } z \text{ values such that } P(z) = 0 \Rightarrow X(z) = 0 \\ \text{poles: all } z \text{ values } \cdots \cdots \; Q(z) = 0 \Rightarrow X(z) \to \infty \end{cases}$$

ROC does not contain poles.

Example:
$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}$$

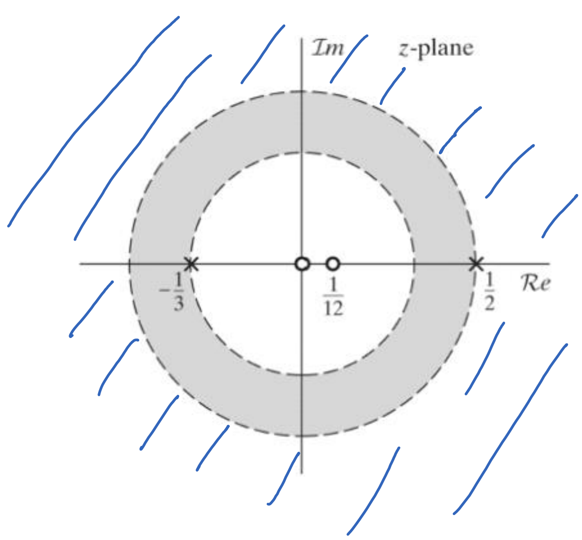
ROC: $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3} \Rightarrow |z| > \frac{1}{2}$

$$X(z) = \frac{1+\frac{1}{3}z^{-1} + 1 - \frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)} \cdot \frac{z^2}{z^2}$$

$$= \frac{\left(2-\frac{1}{6}z^{-1}\right)z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \quad \to 2z\left(z-\frac{1}{12}\right)$$

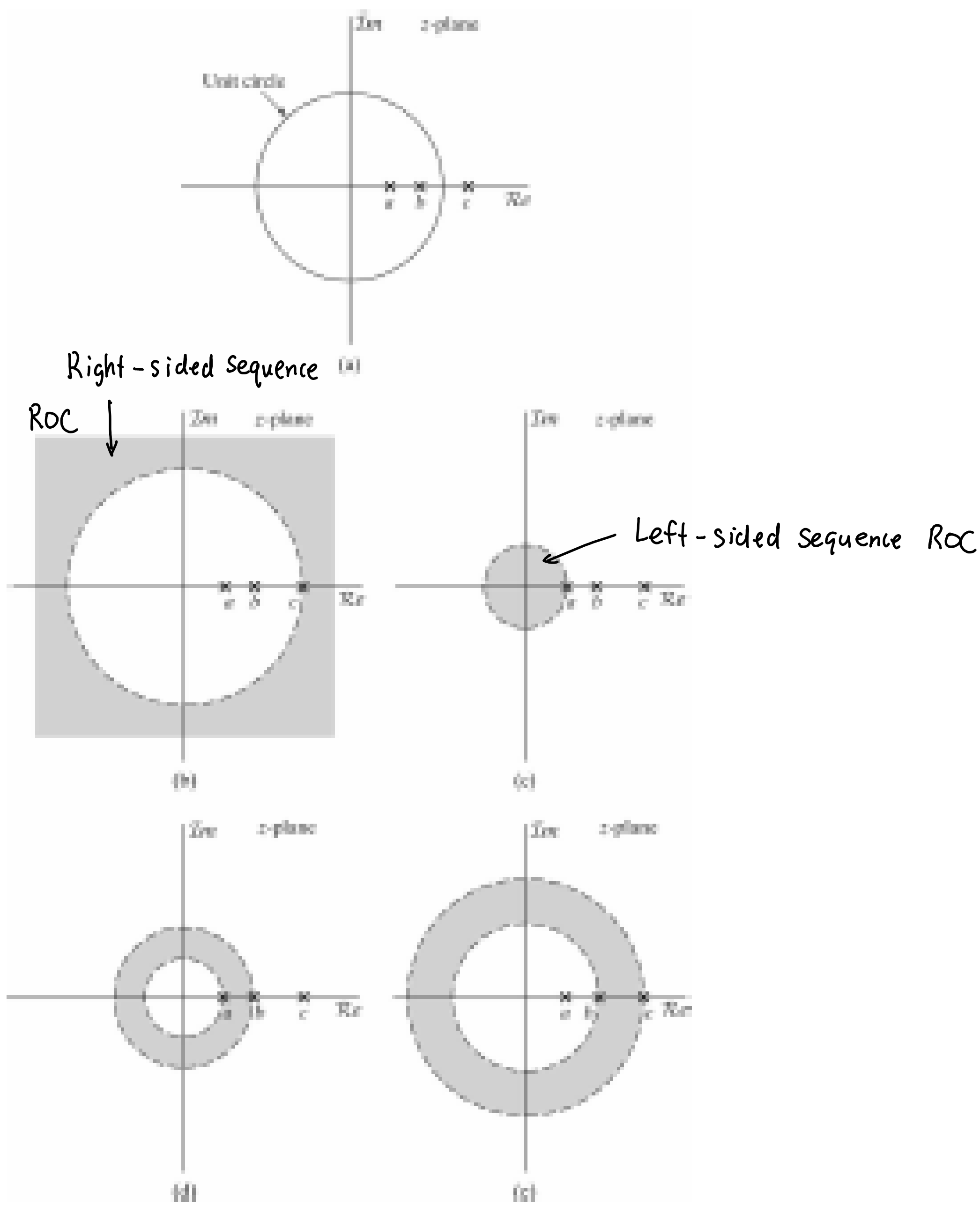
$P(z) = 0 \Rightarrow 2z\left(z-\frac{1}{12}\right) = 0 \Rightarrow$ Two zeros: $0, \frac{1}{12}$.

$Q(z) = 0 \Rightarrow \left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right) = 0 \Rightarrow$ Two poles: $\frac{1}{2}, -\frac{1}{3}$



- Properties of the Z-T
  2: If ROC includes the unit circle, then DTFT of x[n] exists.
  3: ROC cannot contain any poles.
  5: x[n] is right-sided, then ROC extends outward.

  Possibilities of ROC:



- Stability, Causality and ROC

(1) Stability check :

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \Rightarrow \underbrace{\sum_{n=-\infty}^{\infty} h[n] \cdot e^{-j\omega n}}_{\text{DTFT}} \text{ exists} \Rightarrow \text{ROC of } H(z) \text{ includes the unit circle}$$

(2) Causality check

$h[n]$ is right-sided (i.e., $h[n] = 0$ if $n<0$) $\Rightarrow$ causal

$\Rightarrow$ ROC of $H(z)$ extends outward.

Examples : (1) $H(z)$ has ROC : $\frac{1}{2} < |z| < 2$ (ring)

Stable : Yes

Causal : No! Since $h[n]$ is two-sided,

i.e., $h[n] = 0$, if $n<0$

(2) $H(z)$ has a ROC : $|z| > 1$

Stable : No, unit circle not included.

Causal : Yes, ROC extends outward.

**TABLE 3.2** SOME $z$-TRANSFORM PROPERTIES

| Property Number | Section Reference | Sequence | Transform | ROC |
|---|---|---|---|---|
| | | $x[n]$ | $X(z)$ | $R_x$ |
| | | $x_1[n]$ | $X_1(z)$ | $R_{x_1}$ |
| | | $x_2[n]$ | $X_2(z)$ | $R_{x_2}$ |
| 1 | 3.4.1 | $ax_1[n]+bx_2[n]$ | $aX_1(z)+bX_2(z)$ | Contains $R_{x_1}\cap R_{x_2}$ |
| → 2 | 3.4.2 | $x[n-n_0]$ | $z^{-n_0}X(z)$ | $R_x$, except for the possible addition or deletion of the origin or $\infty$ |
| 3 | 3.4.3 | $z_0^n x[n]$ | $X(z/z_0)$ | $\|z_0\|R_x$ |
| 4 | 3.4.4 | $nx[n]$ | $-z\dfrac{dX(z)}{dz}$ | $R_x$ |
| 5 | 3.4.5 | $x^*[n]$ | $X^*(z^*)$ | $R_x$ |
| 6 | | $\mathcal{R}e\{x[n]\}$ | $\frac{1}{2}[X(z)+X^*(z^*)]$ | Contains $R_x$ |
| 7 | | $\mathcal{I}m\{x[n]\}$ | $\frac{1}{2j}[X(z)-X^*(z^*)]$ | Contains $R_x$ |
| 8 | 3.4.6 | $x^*[-n]$ | $X^*(1/z^*)$ | $1/R_x$ |
| → 9 | 3.4.7 | $x_1[n]*x_2[n]$ | $X_1(z)X_2(z)$ | Contains $R_{x_1}\cap R_{x_2}$ |

Time - Shift property : $x[n] \xrightarrow{ZT} X(z)$, $x[n-1] \xrightarrow{ZT} z^{-1}X(z)$

$x[n-2] \xrightarrow{ZT} z^{-2}X(z)$

$\vdots$

Convolution : $y[n] = x[n] * h[n]$

$$Y(z) = X(z)\cdot H(z) \Rightarrow \underset{\uparrow}{H(z)} = \frac{Y(z)}{X(z)}$$

System Function

Recall : $$\underset{\uparrow}{H(e^{j\omega})} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

System Response

For example : $y[n] = x[n] - x[n-1]$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$Y(z) = X(z) - z^{-1}X(z) = (1-z^{-1})X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1}, \quad \text{ROC}: |z| > 0$$

System is both $\begin{cases} \text{stable}: & \text{ROC includes the unit circle ;} \\ \text{causal}: & \text{ROC extends outward.} \end{cases}$