Lecture 7

- Inverse Z-Transforms

(1) Inspection Method

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

(2) Partial Fraction Expansion

$$X(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \quad |z| > \frac{1}{2}.$$

$$= \underbrace{\frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}}$$

$$A_1 = \left[X(z)\left(1-\frac{1}{4}z^{-1}\right)\right]\Bigg|_{z=\frac{1}{4}} = \left[A_1 + \underbrace{\frac{A_2}{1-\frac{1}{2}z^{-1}}\underbrace{\left(1-\frac{1}{4}z^{-1}\right)}_{\Downarrow \atop 0}}_{0}\right]\Bigg|_{z=\frac{1}{4}}$$

Thus

$$A_1 = \left[X(z)\left(1-\frac{1}{4}z^{-1}\right)\right]\Bigg|_{z=\frac{1}{4}}$$

$$= \left[\frac{1}{\cancel{\left(1-\frac{1}{4}z^{-1}\right)}\left(1-\frac{1}{2}z^{-1}\right)}\cancel{\left(1-\frac{1}{4}z^{-1}\right)}\right]\Bigg|_{z=\frac{1}{4}}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}}\Bigg|_{z=\frac{1}{4}}$$

$$= \frac{1}{1-\frac{1}{2}\cdot 4} = -1$$

$$A_2 = \left[X(z)\left(1-\frac{1}{2}z^{-1}\right)\right]\Bigg|_{z=\frac{1}{2}} = \frac{1}{1-\frac{1}{4}z^{-1}}\Bigg|_{z=\frac{1}{2}} = \frac{1}{1-\frac{1}{2}} = 2$$

$$X(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}, \qquad \text{ROC}: |z| > \frac{1}{2}.$$

Check :

$$X(z) = \frac{-1\left(1-\frac{1}{2}z^{-1}\right) + 2\left(1-\frac{1}{4}z^{-1}\right)}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

Thus

$$\frac{-1}{1-\frac{1}{4}z^{-1}} \xrightarrow{Z^{-1}} -\left(\frac{1}{4}\right)^n u[n]$$

$$\frac{2}{1-\frac{1}{2}z^{-1}} \xrightarrow{Z^{-1}} 2\cdot\left(\frac{1}{2}\right)^n u[n]$$

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\cdot\left(\frac{1}{2}\right)^n u[n].$$

We need the ROC to tell

$$\frac{1}{1-az^{-1}} \xrightarrow{Z^{-1}} \begin{cases} a^n u[n], & |z| > |a| \\ -a^n u[-n-1], & |z| < |a| \end{cases}$$

(3) Long Division

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}, \qquad |z| > 1.$$

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \underline{z^{-2} - 3z^{-1} + 2} \\ & \qquad 5z^{-1} - 1 \quad \longleftarrow \text{Remainder} \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$$\Big\downarrow z^{-1}$$

$$x[n] = 2\delta[n] + \cdots$$

$$B(z) = \frac{-1 + 5z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)} = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

$$A_1 = \left[B(z)\left(1 - \frac{1}{2}z^{-1}\right)\right]\Bigg|_{z=\frac{1}{2}} = \frac{-1 + 5z^{-1}}{1 - z^{-1}}\Bigg|_{z=\frac{1}{2}} = \frac{-1 + 5\times 2}{1 - 2} = -9$$

$$A_2 = \left[B(z)\left(1 - z^{-1}\right)\right]\Bigg|_{z=1} = \frac{-1 + 5z^{-1}}{1 - \frac{1}{2}z^{-1}}\Bigg|_{z=1} = \frac{-1+5}{1 - \frac{1}{2}} = 8$$

$$B(z) \xrightarrow{\mathcal{Z}^{-1}} -9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

$$X(z) \xrightarrow{\mathcal{Z}^{-1}} x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

Poles and zeros of $X(z)$

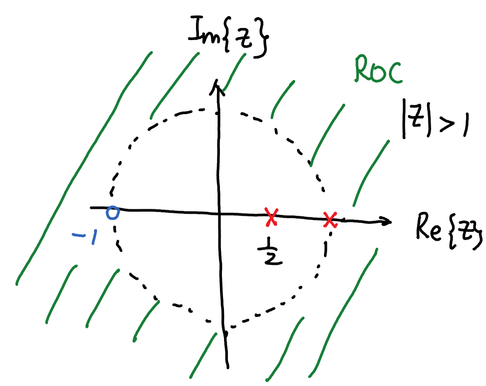
$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}, \quad |z| > 1.$$

$$= \frac{z^2}{z^2} X(z) = \frac{z^2\left(1 + 2z^{-1} + z^{-2}\right)}{z^2\left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{z^2 + 2z + 1}{z^2 - \frac{3}{2}z + \frac{1}{2}}$$

$$= \frac{(z+1)^2}{\left(z - \frac{1}{2}\right)(z-1)}$$

$(z+1)^2 = 0 \Rightarrow$ zeros: -1

$\left(z - \frac{1}{2}\right)(z-1) = 0 \Rightarrow$ poles: $\frac{1}{2}, 1$



(4) Power Series Expansion

Example : $X(z) = \dfrac{1}{1 - az^{-1}}, \quad |z| > |a|$

$$\downarrow$$

$$x[n] = a^n u[n]$$

Long division

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2z^{-2}} \\
 & \qquad a^2z^{-2} \\
 & \qquad \underline{a^2z^{-2} - a^3z^{-3}} \\
 & \qquad\qquad a^3z^{-3} \\
 & \qquad\qquad \vdots
\end{array}
$$

$$X(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$$

Comparing with $X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$

$$X(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots a^nz^{-n} + \cdots$$

$$\cdots \; x[-2] = 0, \quad x[-1] = 0, \quad x[0] = 1, \quad x[1] = a, \quad x[2] = a^2, \quad x[3] = a^3, \quad x[n] = a^n$$

Therefore, $x[n] = a^n u[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

- Finite-length truncated sequence

$$x[n] = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere} \end{cases}$$

Question : $X(z)$?

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(\frac{a}{z}\right)^n$$

$$= \frac{1-\left(\frac{a}{z}\right)^N}{1-\frac{a}{z}} = \frac{1-\frac{a^N}{z^N}}{1-\frac{a}{z}} = \frac{\frac{z^N-a^N}{z^N}}{\frac{z-a}{z}} = \frac{1}{z^{N-1}} \cdot \frac{z^N-a^N}{z-a}$$

Geometric Series, $\displaystyle\sum_{n=0}^{N-1} \beta^n = \frac{1-\beta^N}{1-\beta}$, $(\beta \neq 1)$

For example, $\beta = 2$, $\displaystyle\sum_{n=0}^{\overset{N}{\downarrow}2-1} \beta^n = 2^0 + 2^1 = 3$

$$= \frac{1-2^2}{1-2} = \frac{1-4}{-1} = 3$$

If $N=16$, $0<a<1$

$$X(z) = \frac{1}{z^{15}} \frac{z^{16}-a^{16}}{z-a} = \frac{P(z)}{Q(z)}$$

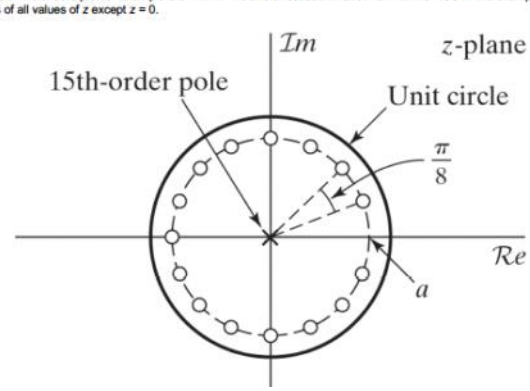
$P(z) = 0 \Rightarrow z^{16} - a^{16} = 0 \Rightarrow z^{16} = a^{16}$, zeros roots: $z = a e^{j\frac{2\pi}{16}k}$

$k = 0, 1, 2, \dots, 15$

$Q(z) = 0 \Rightarrow z^{15}(z-a) = 0 \Rightarrow$ Poles: $z = a$

15 poles at $z = 0$

Figure 3.7 Pole–zero plot for Example 3.6 with $N = 16$ and a real such that $0 < a < 1$. The ROC in this example consists of all values of z except $z = 0$.



Zeros: $k=0$, $z = a e^{j0} = a$

$\downarrow$

Cancelled with $z-a=0$ (pole)

$k=1$, $z = a e^{j\frac{\pi}{8}}$

$k=2$, $z = a e^{j\frac{\pi}{4}}$

$\vdots$

$$X(z) = \frac{1}{z^{15}} \cdot \frac{z^{16}-a^{16}}{z-a}$$

$$= \frac{1}{z^{15}} \cdot \frac{\cancel{(z-a)}\,(z-ae^{j\frac{\pi}{8}})\,(z-ae^{j\frac{\pi}{4}}) \cdots (z-ae^{j\frac{\pi}{16}\cdot 15})}{\cancel{(z-a)}}$$