Lecture 8

Chapter 4 Sampling of Continuous-Time Signals

- Review of Fourier Transforms and Fourier Series

CTFT $X(j\Omega)$	DTFT $X(e^{j\omega})$	Non-Periodic
CTFS	DTFS	Periodic

C-T Signals $x(t)$

D-T sequences $x[n]$

Periodicity:

$x(t) = x(t+T)$ for all t.

$x[n] = x[n+N]$ for all n

- Periodic Signal x(t) = x(t + T)
  We have CTFS of x(t):

$$\begin{cases} x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}, \quad k = \cdots, -2, -1, 0, 1, 2, \cdots \\ a_k = \frac{1}{T}\int_T x(t)\cdot e^{-jk\frac{2\pi}{T}t}\,dt \end{cases}$$

Signal x(t) is periodic in time -> discrete in frequency domain (a_k)

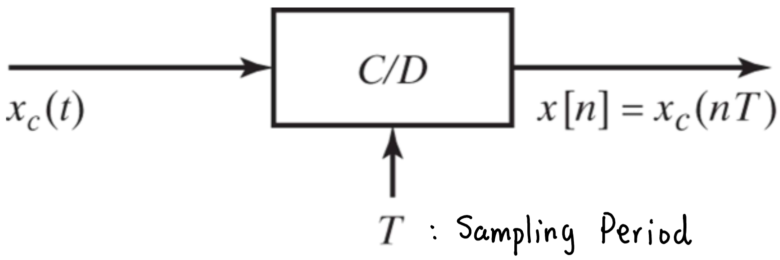
Duality:
Signal x[n] is discrete in time -> DTFT, $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$
periodic in frequency domain

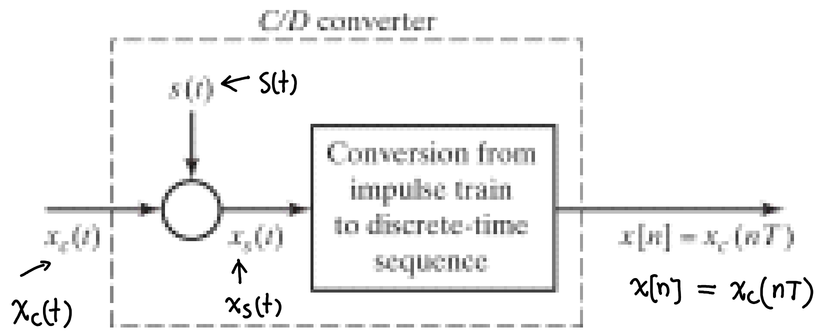
- CTFT (Continuous-Time Fourier Transform)

$$x(t) \xrightarrow{CTFT} X(j\Omega)$$

$$\begin{cases} X(j\Omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\Omega t}\,dt \\ x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\,d\Omega \end{cases}$$

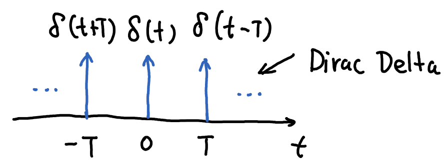
- Sampling (C/D conversion from $x_c(t)$ -> x[n])

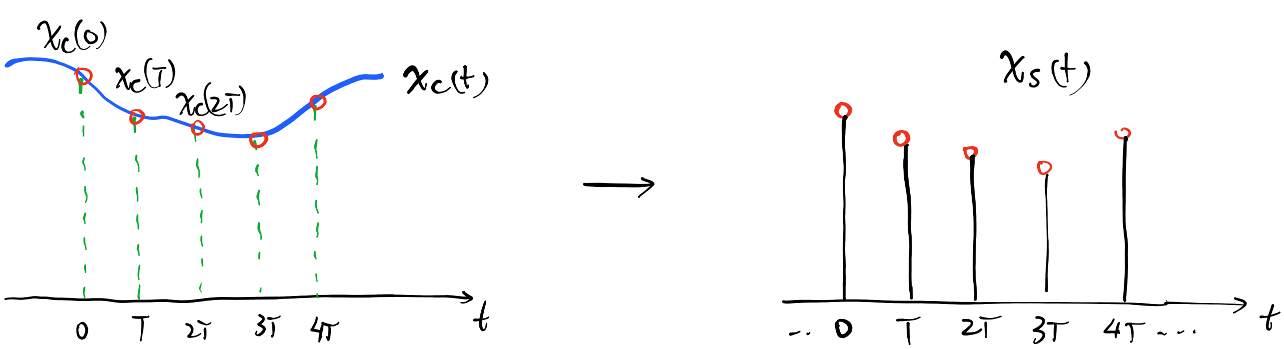


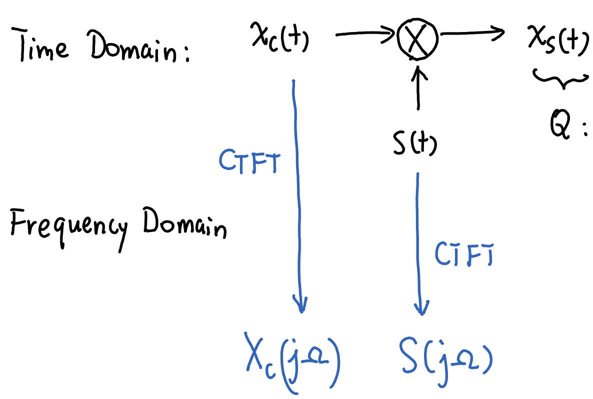


Impulse Train :

$$S(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$







Q : determine $X_s(j\Omega)$?

- Properties of CTFT

$$x(t) \xrightarrow{FT} X(j\Omega)$$

Multiplication Property:

$$\underbrace{x(t)\cdot y(t)}_{\text{multiplication}} \xrightarrow{FT} \frac{1}{2\pi} \underbrace{X(j\Omega) * Y(j\Omega)}_{\text{Convolution Integral}}$$

Differential Property:

$$y(t) = \frac{dx(t)}{dt} \xrightarrow{FT} Y(j\Omega) = j\Omega X(j\Omega)$$

- Determine $S(j\Omega)$ given $s(t)$.

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \cdots \delta(t+T) + \delta(t) + \delta(t-T) + \cdots$$

$s(t)$ is periodic with T being the period.

$$s(t) \xrightarrow{CTFT} S(j\Omega)\ ?$$

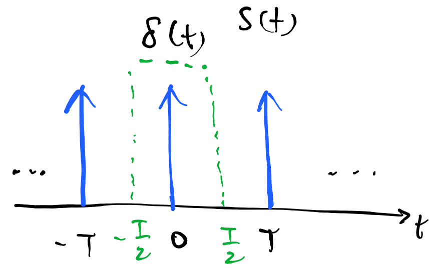
Approach: $s(t) \xrightarrow{CTFS} a_k \longrightarrow$ CTFT of $s(t)$.

$s(t) = s(t+T)$

CTFS: $$a_k = \frac{1}{T}\int_T s(t)\cdot e^{-jk\frac{2\pi}{T}t}\,dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\cdot e^{-jk\frac{2\pi}{T}}dt$$

$$= \frac{1}{T}$$



Thus $$s(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\frac{2\pi}{T}t} = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$$

since $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}, \quad k = \cdots, -2, -1, 0, 1, 2, \cdots$$

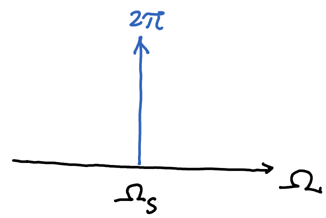
$$s(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$$

Next, find out the CTFT of $e^{jk\frac{2\pi}{T}t}$

Consider $X(j\Omega) = 2\pi\,\delta(\Omega - \Omega_s)$

$$\Big\downarrow \mathcal{F}_T^{-1}$$

$x(t)$?



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\Omega - \Omega_s)\, e^{j\Omega t}\, d\Omega$$

$$= e^{j\Omega_s t}$$

$$= \cos(\Omega_s t) + j\sin(\Omega_s t)$$

In summary,

$$e^{j\Omega_s t} \xrightarrow{\text{CTFT}} 2\pi\,\delta(\Omega - \Omega_s)$$

$$\downarrow$$

$$a_k e^{jk\frac{2\pi}{T}t} \xrightarrow{\text{CTFT}} a_k \cdot 2\pi\,\delta\left(\Omega - k\cdot\frac{2\pi}{T}\right)$$

Thus $s(t) \xrightarrow{\mathcal{F}_T} S(j\Omega) = a_k \cdot 2\pi \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\cdot\frac{2\pi}{T}\right)$

$$a_k \to \frac{1}{T}$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\cdot\frac{2\pi}{T}\right)$$