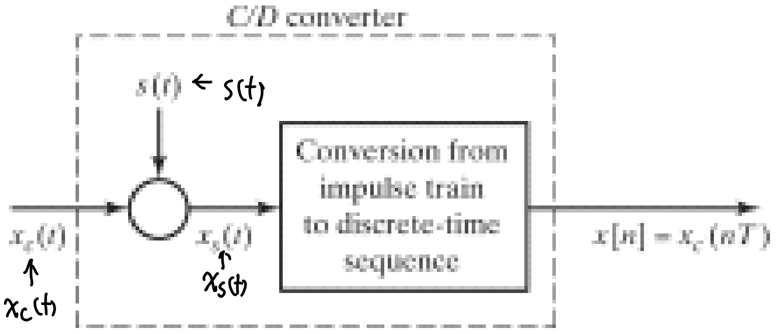
Lecture 9



$$\begin{tikzcd}
x_c(t) \arrow[r] \arrow[dd, "\text{CTFT}"'] & \otimes \arrow[r] & x_s(t) \arrow[dd, "\text{CTFT}"] \\
 & s(t) \arrow[u] \arrow[d, "\text{CTFT}"] & \\
X_c(j\Omega) & S(j\Omega) & X_s(j\Omega)
\end{tikzcd}$$

Time Domain: $x_s(t) = x_c(t) \underset{\substack{\uparrow \\ \text{multiplication}}}{\cdot} s(t)$

Frequency Domain

$$X_s(j\Omega) = \frac{1}{2\pi}\left[X_c(j\Omega) \underset{\substack{\uparrow \\ \text{Convolution}}}{*} S(j\Omega)\right]$$

$X_c(j\Omega)$ ↓ given

$S(j\Omega)$ ↓

$$S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\Omega - k\frac{2\pi}{T}\right)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\underset{\sim}{\Omega_s})$$

where $\Omega_s = \frac{2\pi}{T}$ ← Sampling period (s)

Sampling frequency (rad/s)

$$X_s(j\Omega) = \frac{1}{2\pi}\cdot\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\left[X_c(j\Omega) * \delta(\Omega - k\Omega_s)\right]$$

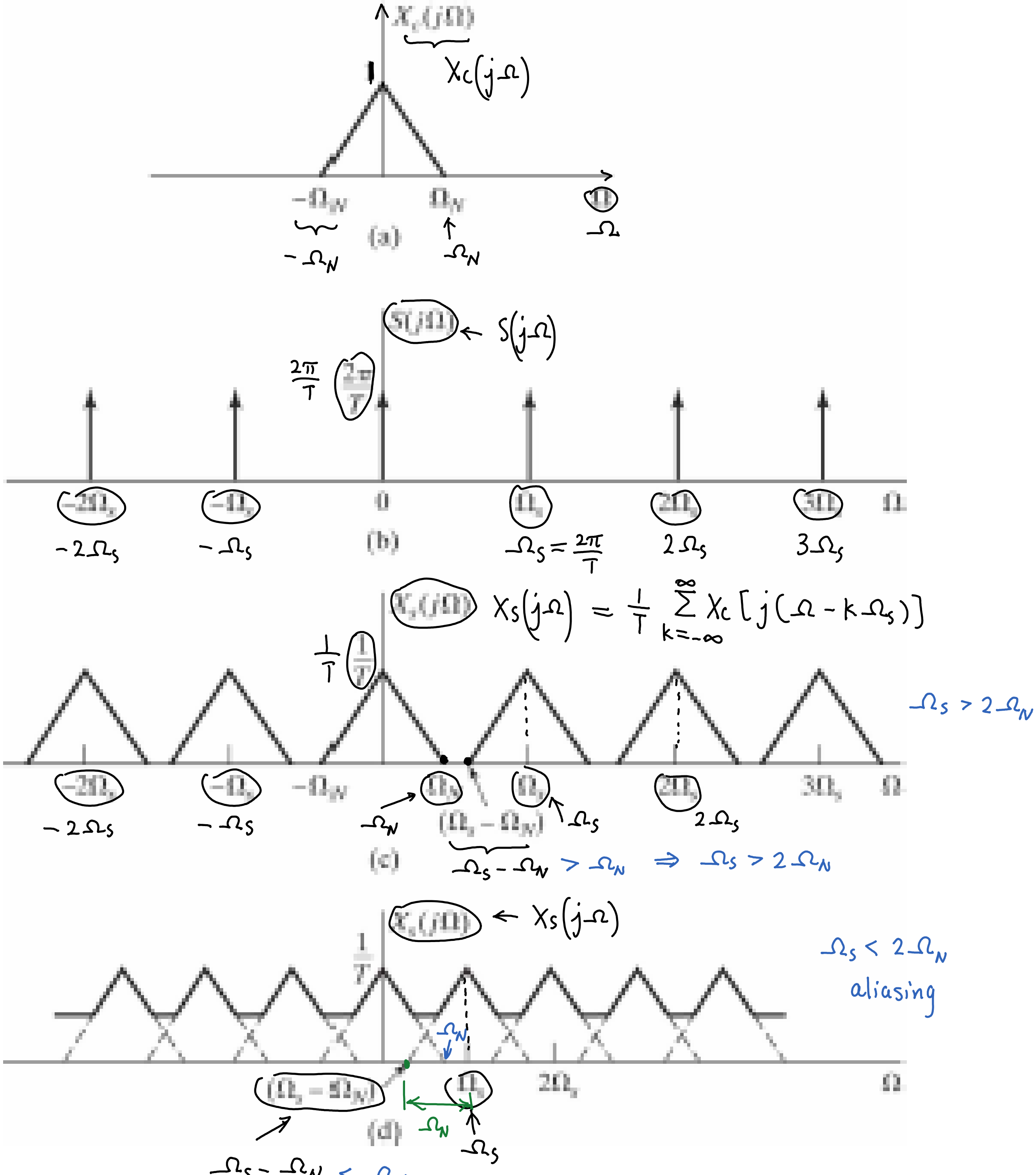
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j(\Omega - k\cdot\Omega_s)\right]$$

Since

$k=0$, $X_c(j\Omega) * \delta(\Omega - 0\Omega_s) = X_c(j\Omega) * \delta(\Omega) = X_c\left[j(\Omega - 0\cdot\Omega_s)\right]$

$k=1$, $X_c(j\Omega) * \delta(\Omega - 1\Omega_s) = X_c\left[j(\Omega - 1\cdot\Omega_s)\right]$

⋮



$\Omega_s = \frac{2\pi}{T}$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c[j(\Omega - k\Omega_s)]$$

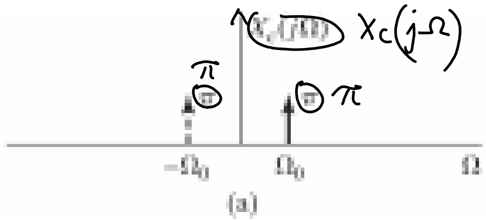
$\Omega_s - \Omega_N > \Omega_N \Rightarrow \Omega_s > 2\Omega_N$

$\Omega_s - \Omega_N < \Omega_N \Rightarrow \Omega_s < 2\Omega_N$

Nyquist Theorem: $\Omega_s \geqslant 2\Omega_N$. perfect reconstruction of $x_c(t)$.

- Example of aliasing

$$\cos(\Omega_0 t) = \frac{1}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)$$

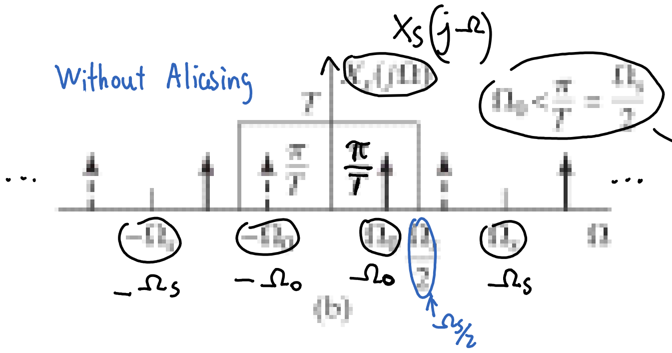


$$e^{j\Omega_0 t} \xrightarrow{CTFT} 2\pi\,\delta(\Omega - \Omega_0)$$

$$\underbrace{FT\{\cos(\Omega_0 t)\}}_{X_c(j\Omega)} = \frac{1}{2}\left[FT\{e^{j\Omega_0 t}\} + FT\{e^{-j\Omega_0 t}\}\right]$$

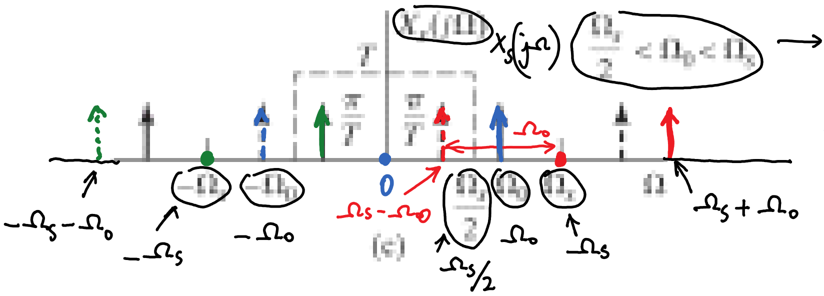
$$= \frac{1}{2}\left[2\pi\,\delta(\Omega - \Omega_0) + 2\pi\,\delta(\Omega + \Omega_0)\right]$$

$$= \pi\,\delta(\Omega - \Omega_0) + \pi\,\delta(\Omega - (-\Omega_0))$$



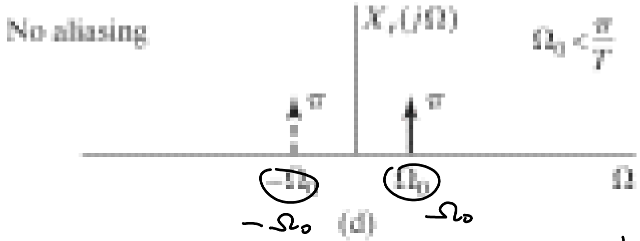
$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c[j(\Omega - k\Omega_s)]$$

$$\rightarrow \Omega_s > 2\Omega_0 \Rightarrow \Omega_0 < \frac{\Omega_s}{2} = \frac{\frac{2\pi}{T}}{2} = \frac{\pi}{T}$$

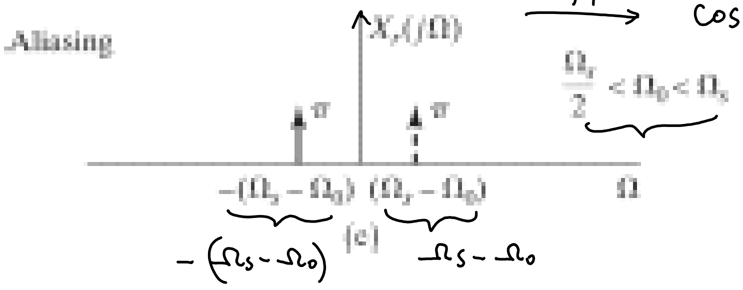


$$\rightarrow \frac{\Omega_s}{2} < \Omega_0 \Rightarrow \Omega_s < 2\Omega_0$$

Aliasing



$$\xrightarrow{FT^{-1}} \cos[(\Omega_s - \Omega_0)t] \neq \cos(\Omega_0 t)$$



$$\frac{\Omega_s}{2} < \Omega_0 < \Omega_s$$

$$\Rightarrow \Omega_s < 2\Omega_0$$