The University of Alabama in Huntsville
Electrical and Computer Engineering
CPE 633 01
Homework #1 Solution
Spring 2008

1. The lifetime (measured in years) of a processor is exponentially distributed, with a mean lifetime of 2 years. You are told that a processor failed sometime in the interval [4, 8] years. Given this information, what is the conditional probability that it failed before it was 5 years old?

   Denote the lifetime of the processor by T. Since E(T) = 2, $\lambda = 0.5$ and the distribution function of T is $F(t) = 1 - e^{-0.5\lambda t}$. Using the conditional probability formula:
   Prob{T < 5 | 4 ≤ T ≤ 8} =

$$\frac{\text{Prob}\{T<5\} \cap \text{Prob}\{4 \le T < 8\}}{\text{Prob}\{4 \le T < 8\}} = \frac{\text{Prob}\{4 \le T < 5\}}{\text{Prob}\{4 \le T < 8\}} = \frac{F(5)-F(4)}{F(8)-F(4)} = \frac{(1-e^{-5(0.5)})-(1-e^{-4(0.5)})}{(1-e^{-8(0.5)})-(1-e^{-4(0.5)})}$$

$$= \frac{e^{-2}-e^{-2.5}}{e^{-2}-e^{-4}} = 0.455$$

2. The lifetime of a processor (measured in years) follows the Weibull distribution, with parameters λ = 0.5 and β = 0.6.
   (a) What is the probability that it will fail in its first year of operation?
   (b) Suppose it is still functional after t = 6 years of operation. What is the conditional probability that it will fail in the next year?
   (c) Repeat parts (a) and (b) for β = 2.
   (d) Repeat parts (a) and (b) for β = 1.

   (a) Denote the lifetime of the processor by T. The distribution function of T is $F(t) = 1 - e^{-0.5t^{0.6}}$. The probability that T is no greater than one year is

$$F(1) = 1 - e^{(-0.5)1^{0.6}} = 1 - e^{-0.5} = 1 - 0.606 = 0.394$$

   (b) We use the conditional probability formula:

$$\frac{\text{Prob}\{T<7\} \cap \text{Prob}\{T>6\}}{\text{Prob}\{T>6\}} = \frac{\text{Prob}\{6 \le T < 7\}}{1 - F(6)} = \frac{F(7)-F(6)}{1-1-e^{(-0.5)6^{0.6}}} = \frac{(1-e^{-(0.5)7^{0.6}})-(1-e^{-(0.5)6^{0.6}})}{e^{(-0.5)6^{0.6}}}$$

$$= \frac{e^{-(0.5)6^{0.6}}-e^{-(0.5)7^{0.6}}}{e^{(-0.5)6^{0.6}}} = \frac{0.231-0.200}{0.231} 0.134$$

   (c) $F(1) = 1 - e^{(-0.5)1^{2}} = 1 - e^{-0.5} = 1 - 0.606 = 0.394$

$$\frac{\text{Prob}\{T<7\} \cap \text{Prob}\{T>6\}}{\text{Prob}\{T>6\}} = \frac{\text{Prob}\{6 \le T < 7\}}{1 - F(6)} = \frac{F(7)-F(6)}{1-1-e^{(-0.5)6^{2}}} = \frac{(1-e^{-(0.5)7^{2}})-(1-e^{-(0.5)6^{2}})}{e^{(-0.5)6^{2}}}$$

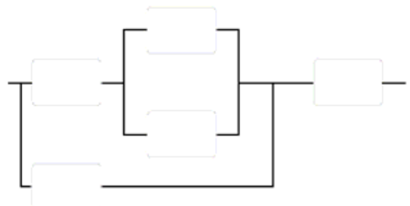
$$= \frac{e^{-(0.5)6^{2}}-e^{-(0.5)7^{2}}}{e^{(-0.5)6^{0.6}}} = \frac{1.52\times10^{-8}-2.29\times10^{-11}}{1.52\times10^{-8}} = 1.00$$

(d) $F(1) = 1 - e^{(-0.5)1^1} = 1 - e^{-0.5} = 1 - 0.606 = 0.394$

$$\frac{\text{Prob}\{T<7\} \cap \text{Prob}\{T>6\}}{\text{Prob}\{T>6\}} = \frac{\text{Prob}\{6 \le T < 7\}}{1 - F(6)} = \frac{F(7) - F(6)}{1 - 1 - e^{(-0.5)6^1}} = \frac{(1 - e^{-(0.5)7^1}) - (1 - e^{-(0.5)6^1})}{e^{(-0.5)6^1}}$$

$$= \frac{e^{-(0.5)6^1} - e^{-(0.5)7^1}}{e^{(-0.5)6^1}} = \frac{0.0498 - 0.0302}{0.0498} = 0.394$$

4. Write the expression for the reliability $R_{system}(t)$ of the series/parallel system shown in Figure 2.2, assuming that each of the five modules has a reliability of R(t).



The system can be decomposed into a series system consisting of one unit with the leftmost 4 blocks and the second unit with the rightmost block. If the reliability of the leftmost 4 blocks is $R_A(t)$, the system reliability is $R_A(t)R(t)$. Now, we calculate $R_A(t)$. This subsystem consists of a parallel arrangement of one unit consisting of the bottom block and another consisting of the other 3 blocks. If $R_B(t)$ is the reliability of the top 3 blocks, $R_A(t) = 1 - (1 - R_B(t))(1 - R(t))$. Next, we calculate $R_B(t)$: this subsystem consists of a series arrangement of one block with another consisting of two blocks in parallel. Hence, we have

$R_B(t) = R(t)(1 - (1 - R(t))^2) = R(t)(1 – (1 – 2R(t) + R^2(t))) = R(t)((2R(t) – R^2(t)) = 2R^2(t) – R^3(t)$

$R_A(t) = 1 – (1 – (2R^2(t) – R^3(t)))(1 – R(t))$
$= 1 – (1 – 2R^2(t) + R^3(t))(1 – R(t))$
$= 1 – (1 – R(t) – 2R^2(t) + 2R^3(t) + R^3(t) – R^4(t)$
$= R(t) + 2R^2(t) – 3R^3(t) + R^4(t)$

$R_{system} = R_A(t)R(t) = R^5(t) – 3R^4(t) + 2R^3(t) + R^2(t)$

12. Consider a system consisting of 2 subsystems in series. For improved reliability, you can build subsystem i as a parallel system with ki units, for i = 1, 2. Suppose permanent failures occur at a constant rate λ per unit.
(a) Derive an expression for the reliability of this system.
(b) Obtain an expression for the MTTF of this system with $k_1 = 2$ and $k_2 = 3$.
(a) The reliability of a parallel system with k units is $R_p^{(k)}(t) = 1 - (1 - e^{-\lambda t})^k$. Hence, the releiability of the series system is given by $R_{series} = R_p^{(k_1)}(t) R_p^{(k_2)}(t)$
(b) For $k_1 = 2$ and $k_2 = 3$,

$$R_p^{(k_1)}(t) = 1 - (1 - e^{-\lambda t})^{k_1} = R_p^{(2)}(t) = 1 - (1 - e^{-\lambda t})^2 = 1 - (1 - 2e^{-\lambda t} + e^{-2\lambda t}) = 2e^{-\lambda t} - e^{-2\lambda t}$$
$$R_p^{(k_2)}(t) = 1 - (1 - e^{-\lambda t})^{k_2} = R_p^{(3)}(t) = 1 - (1 - e^{-\lambda t})^3 = 1 - ((1 - 2e^{-\lambda t} + e^{-2\lambda t})(1 - e^{-\lambda t}))$$
$$= 1 - (1 - e^{-\lambda t} - 2e^{-\lambda t} + 2e^{-2\lambda t} + e^{-2\lambda t} - e^{-3\lambda t}) = 3e^{-\lambda t} - 3e^{-2\lambda t} + e^{-3\lambda t}$$
$$R_{series} = (2e^{-\lambda t} - e^{-2\lambda t})(3e^{-\lambda t} - 3e^{-2\lambda t} + e^{-3\lambda t}) = 6e^{-2\lambda t} - 6e^{-3\lambda t} + 2e^{-4\lambda t} - 3e^{-3\lambda t} + 3e^{-4\lambda t} - e^{-5\lambda t}$$

$$= 6e^{-2\lambda t} - 9e^{-3\lambda t} + 5e^{-4\lambda t} - e^{-5\lambda t}$$

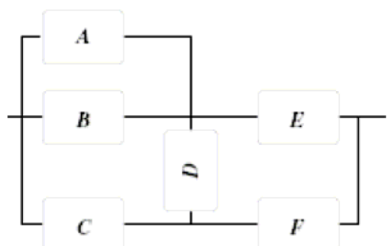
$$MTTF = \int_{t=0}^{\infty} R_{series}(t)dt = \int_{t=0}^{\infty} (6e^{-2\lambda t} - 9e^{-3\lambda t} + 5e^{-4\lambda t} - e^{-5\lambda t})(t)dt$$

$$= \frac{6}{-2\lambda}e^{-2\lambda t} - \frac{9}{-3\lambda}e^{-3\lambda t} + \frac{5}{-4\lambda}e^{-4\lambda t} - \frac{1}{-5\lambda}e^{-5\lambda t}\Big|_{t=0}^{\infty}$$

$$= \left[\frac{6}{-2\lambda}e^{-2\lambda\infty} - \frac{9}{-3\lambda}e^{-3\lambda\infty} + \frac{5}{-4\lambda}e^{-4\lambda\infty} - \frac{1}{-5\lambda}e^{-5\lambda\infty}\right] - \left[\frac{6}{-2\lambda}e^{-2\lambda 0} - \frac{9}{-3\lambda}e^{-3\lambda 0} + \frac{5}{-4\lambda}e^{-4\lambda 0} - \frac{1}{-5\lambda}e^{-5\lambda 0}\right]$$

$$= [0 - 0 + 0 - 0] - \left[-\frac{6}{2\lambda} + \frac{9}{3\lambda} - \frac{5}{4\lambda} + \frac{1}{5\lambda}\right] = \frac{3}{\lambda} - \frac{3}{\lambda} + \frac{5}{4\lambda} - \frac{1}{5\lambda} = \frac{25-4}{20\lambda} = \frac{21}{20\lambda}$$

14. Write expressions for the upper and lower bounds and the exact reliability of the non series/parallel system shown in Figure 2.6 (denote by $R_i(t)$ the reliability of module i). Assume that D is a bi-directional unit.



The paths are: AE, BE, CF, ADF, BDF and CDE.

The upper bound is

$R_{system} \leq 1 - (1 - R_A R_E)(1 - R_B R_E)(1 - R_C R_F)(1 - R_A R_D R_F)(1 - R_B R_D R_F)(1 - R_C R_D R_E)$

The minimal cut sets are: EF, ABC, CDE and ABDF. The lower bound is

$R_{system} \geq [1-(1-R_E)(1-R_F)][1-(1-R_A)(1-R_B)(1-R_C)]\ [1-(1-R_C)(1-R_D)(1-R_E)][1-(1-R_A)(1- R_B)(1-R_D)(1-R_F)]$

To calculate the exact reliability we expand about module D. If D is faulty A and B are connected in parallel, and then in series with E and all these are in parallel to the series connection of C and F yielding

Rsys| D is faulty = $1 - [1 - (R_A + R_B - R_A R_B)R_E](1 - R_C R_F)$

If D is fault-free, A, B and C are connected in parallel, and then in series with the parallel connection of E and F yielding

Rsys|D is fault-free = $[1 - (1 - R_A)(1 - R_B)(1 - R_C)][R_E + R_F - R_E R_F]$

Finally, $R_{system} = R_D$Rsys|D is fault-free + $(1 - R_D)$Rsys|D is fault

$= R_D[1-(1 - R_A)(1 - R_B)(1 - R_C)][R_E + R_F - R_E R_F] + (1 - R_D)\ 1 - [1 - (R_A + R_B - R_A R_B)R_E](1 - R_C R_F)$