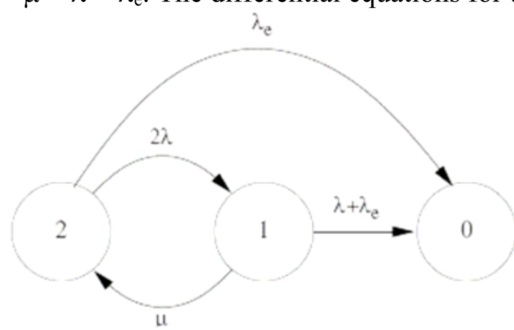
The University of Alabama in Huntsville
Electrical and Computer Engineering
Homework #4 Solution
CPE 633 01
Spring 2008

Chapter 3: Problems 12(15 points), 14(30 points), 15(15 points), 18(30 points), 21(10 points)

12. You have a RAID1 system where failures occur at individual disks at a constant rate λ per disk. The repair time of disks is exponentially distributed with rate μ. Suppose we are in an earthquake-prone area, where building-destroying earthquakes occur according to a Poisson process with rate λ_e. If the building is destroyed, so too is the entire RAID system. Derive an expression for the probability of data loss for such a system as a function of time. Assuming that the mean time between such earthquakes is 50 years, plot the probability of data loss as a function of time using the parameters 1/λ = 500,000 hours and 1/μ = 1 hour.

    Modify to only generate Markov chain and differential equations.

    Define the state of the system by how many disks are up for i = 1, 2 if data loss has not occurred, and 0 if it has. $\pi_i(t)$ is the probability of being in state i at time t, we seek $1 - \pi_0(t)$. For convenience, define $a = \lambda_e + 2\lambda$, $b = \mu$, $c = 2\lambda$, $d = \mu + \lambda + \lambda_e$. The differential equations for the Markov chain are:



$$\frac{d\pi_2(t)}{dt} = -a\pi_2(t) + b\pi_1(t)$$
$$\frac{d\pi_1(t)}{dt} = c\pi_2(t) - d\pi_1(t)$$

    with the initial conditions $\pi_2(0) = 1$, $\pi_1(0) = 0$.

14. Given a RAID level 5 system with an orthogonal arrangment of d + 1 strings and g = 8 RAID groups, compare the MTTDL for different values of d from 4 to 10. Assume an exponential repair process for single disks and for strings of disks with repair rates of 1/hour and 3/hour, respectively. Also assume failure rates for single disks and strings of disks of 10^{-6}/hour and 5 x 10^{-6}/hour, respectively.

    The repair rate for disks is $f_d = e^{-t}$ and the repair rate for strings is $f_s = 3e^{-3t}$. $\lambda_d = 10^{-6}$ and $\lambda_s = 5*10^{-6}$.

$$\pi_{indiv} = \int_0^{\infty} (1 - e^{-d(\lambda_{disk} + \lambda_{str})^{\tau}} f_{disk}(\tau) d\tau$$

$$\pi_{indiv} = \int_0^\infty (1 - e^{-d(\lambda_{disk} + \lambda_{str})\tau}) e^{-\tau} d\tau$$

$$\pi_{indiv} = \int_0^\infty e^{-\tau} d\tau - \int_0^\infty e^{(-d(10^{-6} + 5*10^{-6}) + 1)\tau} d\tau$$

$$\pi_{indiv} = \left[-e^{-t} \right]_0^\infty - \left[\frac{1}{-d(6*10^{-6}) + 1} e^{(-d(6*10^{-6})+1)t} \right]_0^\infty$$

$$\pi_{indiv} = \left[-e^{-\infty} - (-e^{-0}) \right] - \left[\frac{1}{-d(6*10^{-6}) + 1} e^{(-d(6*10^{-6})+1)\infty} - \frac{1}{-d(6*10^{-6}) + 1} e^{(-d(6*10^{-6})+1)0} \right]$$

$$\pi_{indiv} = \left[-0 - (-1) \right] - \left[-0 - \left(\frac{1}{-d(6*10^{-6}) + 1} \right) \right]$$

$$\pi_{indiv} = 1 - \left(\frac{1}{d(6*10^{-6}) + 1} \right)$$

$$\pi_{pess} = \int_0^\infty (1 - e^{-(d+1)(g\lambda_{disk} + \lambda_{str})\tau}) f_{str}(\tau) d\tau$$

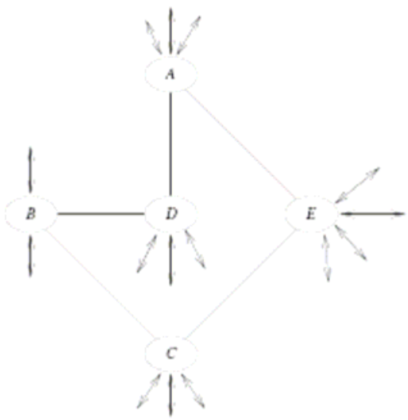
$$\pi_{pess} = \int_0^\infty (1 - e^{-(d+1)(g\lambda_{disk} + \lambda_{str})\tau}) 3e^{-3\tau} d\tau$$

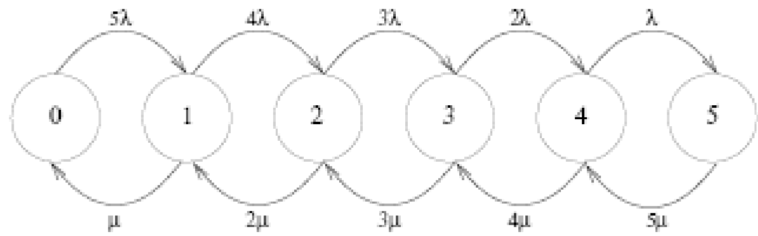
$$\pi_{pess} = 1 - \left(\frac{3}{(d+1)(13*10^{-6}) + 3} \right)$$

$$\pi_{opt} = 1 - \left(\frac{3}{d(13*10^{-6}) + 3} \right)$$

d	piindiv	lindiv	lpess	pipess	lstrpess	lopt	piopt	lstropt	ldloss-p	MTTDL-p	ldloss-o	MTTDL-o
4	2.40E-05	9.60E-10	6.50E-05	2.17E-05	5.42E-10	5.20E-05	1.73E-05	4.33E-10	1.50E-09	6.66E+08	1.39E-09	7.18E+08
5	3.00E-05	1.44E-09	7.80E-05	2.60E-05	7.80E-10	6.50E-05	2.17E-05	6.50E-10	2.22E-09	4.50E+08	2.09E-09	4.78E+08
6	3.60E-05	2.02E-09	9.10E-05	3.03E-05	1.06E-09	7.80E-05	2.60E-05	9.10E-10	3.08E-09	3.25E+08	2.93E-09	3.42E+08
7	4.20E-05	2.69E-09	1.04E-04	3.47E-05	1.39E-09	9.10E-05	3.03E-05	1.21E-09	4.07E-09	2.45E+08	3.90E-09	2.56E+08
8	4.80E-05	3.46E-09	1.17E-04	3.90E-05	1.75E-09	1.04E-04	3.47E-05	1.56E-09	5.21E-09	1.92E+08	5.02E-09	1.99E+08
9	5.40E-05	4.32E-09	1.30E-04	4.33E-05	2.17E-09	1.17E-04	3.90E-05	1.95E-09	6.49E-09	1.54E+08	6.27E-09	1.59E+08
10	6.00E-05	5.28E-09	1.43E-04	4.77E-05	2.62E-09	1.30E-04	4.33E-05	2.38E-09	7.90E-09	1.27E+08	7.66E-09	1.30E+08

15. Derive expressions for the reliability and availability of the network shown in Figure 3.3 for the case (r,w) = (3,3) where a single vote is assigned to each node in the nonhierarchical organization. In this case, both read and write operations can take place if at least three of the five nodes are up. Assume that failures occur at each node according to a Poisson process with rate λ, but the links do not fail. When a node fails, it is repaired (repair includes loading up-to-date data) and the repair time is an exponentially distributed random variable with mean $1/\mu$. Derive the required expressions for the system reliability and availability using the Markov chains (see Chapter 2) shown in Figure 3.4a and b, respectively, where the state is the number of nodes that are down.





Modify to do availability only. Availability is modeled by the probability of being in one of the states in the following set: {0, 1, 2} for $t \to \infty$.

$$\frac{dP_0}{dt} = -5\lambda P_0 + \mu P_1 \qquad 0 = -5\lambda P_0 + \mu P_1 \qquad P_1 = \frac{5\lambda}{\mu} P_0$$

$$\frac{dP_1}{dt} = -(4\lambda + \mu)P_1 + 2\mu P_2 + 5\lambda P_0 \qquad 0 = -(4\lambda + \mu)\frac{5\lambda}{\mu} P_0 + 2\mu P_2 + 5\lambda P_0$$

$$0 = -\frac{20\lambda}{\mu^2} P_0 - 5\lambda P_0 + 2\mu P_2 + 5\lambda P_0 \qquad \frac{20\lambda}{\mu^2} P_0 = 2\mu P_2 \qquad P_2 = \frac{10\lambda^2}{\mu^2} P_0$$

$$\frac{dP_2}{dt} = -(3\lambda + 2\mu)P_2 + 3\mu P_3 + 4\lambda P_1 \qquad 0 = -(3\lambda + 2\mu)\frac{10\lambda^2}{\mu^2} P_0 + 3\mu P_3 + \frac{5\lambda}{\mu} P_0$$

$$0 = -\frac{30\lambda^3}{\mu^2} P_0 - \frac{20\lambda^2}{\mu} P_0 + 3\mu P_3 + \frac{20\lambda^2}{\mu} P_0 \qquad \frac{30\lambda^3}{\mu^2} P_0 = 3\mu P_3$$

$$P_3 = \frac{10\lambda^3}{\mu^3} P_0$$

$$\frac{dP_3}{dt} = -(2\lambda + 3\mu)P_3 + 4\mu P_4 + 3\lambda P_2 \qquad 0 = -(2\lambda + 3\mu)\frac{10\lambda^3}{\mu^3} P_0 + 4\mu P_4 + 3\lambda \frac{10\lambda^2}{\mu^2} P_0$$

$$0 = -\frac{20\lambda^4}{\mu^3} P_0 - \frac{30\lambda^3}{\mu^2} P_0 + 3\mu P_3 + \frac{30\lambda^3}{\mu^2} P_0 \qquad \frac{20\lambda^4}{\mu^3} P_0 = 4\mu P_4$$

$$P_4 = \frac{5\lambda^4}{\mu^4} P_0$$

$$\frac{dP_4}{dt} = -(\lambda + 4\mu)P_4 + 5\mu P_5 + 2\lambda P_3 \qquad 0 = -(\lambda + 4\mu)\frac{5\lambda^4}{\mu^4} P_0 + 5\mu P_5 + 2\lambda \frac{10\lambda^3}{\mu^3} P_0$$

$$0 = -\frac{5\lambda^5}{\mu^4} P_0 - \frac{20\lambda^4}{\mu^3} P_0 + 5\mu P_5 + \frac{20\lambda^4}{\mu^3} P_0 \qquad \frac{5\lambda^5}{\mu^4} P_0 = 5\mu P_5$$

$$P_5 = \frac{\lambda^5}{\mu^5} P_0$$

$$\sum_{i=0}^{5} p_i = 1 \qquad p_0\left(1+\frac{5\lambda}{\mu}+\frac{10\lambda^2}{\mu^2}+\frac{10\lambda^3}{\mu^3}+\frac{5\lambda^4}{\mu^4}+\frac{\lambda^5}{\mu^5}\right)=1$$

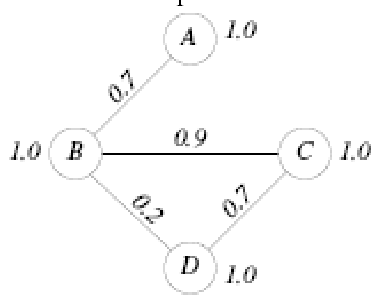
$$p_0 = \left(1+\frac{5\lambda}{\mu}+\frac{10\lambda^2}{\mu^2}+\frac{10\lambda^3}{\mu^3}+\frac{5\lambda^4}{\mu^4}+\frac{\lambda^5}{\mu^5}\right)^{-1}$$

$$p_1 = \frac{5\lambda}{\mu} p_0 = \frac{5\lambda}{\mu}\left(1+\frac{5\lambda}{\mu}+\frac{10\lambda^2}{\mu^2}+\frac{10\lambda^3}{\mu^3}+\frac{5\lambda^4}{\mu^4}+\frac{\lambda^5}{\mu^5}\right)^{-1}$$

$$p_2 = \frac{10\lambda^2}{\mu^2} p_0 = \frac{10\lambda^2}{\mu^2}\left(1+\frac{5\lambda}{\mu}+\frac{10\lambda^2}{\mu^2}+\frac{10\lambda^3}{\mu^3}+\frac{5\lambda^4}{\mu^4}+\frac{\lambda^5}{\mu^5}\right)^{-1}$$

$$A = p_0 + p_1 + p_2 = \left(1+\frac{5\lambda}{\mu}+\frac{10\lambda^2}{\mu^2}\right)\left(1+\frac{5\lambda}{\mu}+\frac{10\lambda^2}{\mu^2}+\frac{10\lambda^3}{\mu^3}+\frac{5\lambda^4}{\mu^4}+\frac{\lambda^5}{\mu^5}\right)^{-1}$$

18. For the example shown in Figure 3.5 the four nodes have an availability 1 while the links have the availabilities indicated in the figure. Use Heuristic 2 to assign votes to the four nodes, write down the possible values for w and the corresponding minimal values of r, and calculate the availability for each possible value of (r,w). Assume that read operations are twice as frequent as write operations



Read and write quorums under Heuristic 2:

v(A) = 1.0 + (0.7*1.0) = 1.7 — Assign 2 votes to A
v(B) = 1.0 + (0.7*1.0 + 0.9*1.0 + 0.1*1.0) = 2.8 — Assign 3 votes to B (now 4 votes)
v(C) = 1.0 + (0.9*1.0 + 0.7*1.0) = 2.6 — Assign 3 votes to C
v(D) = 1.0 + (0.2*1.0 + 0.7*1.0) = 1.9 — Assign 2 votes to D

Total number of votes = 2 + 3 + 3 + 2 = 10. Since 10 is even, add extra vote to one of the largest, pick B. w > 11/2, r + w > 11

R	W	Read Quorums	Write Quorums	Pr	Pw	A
6	6	AB, BC, BD, ACD	AB, BC, BD, ACD	0.976	0.9760	0.9760
5	7	AB, AC, BC, CD, BD	BC, ABD, ACD	0.993	0.9182	0.9679
4	8	B, AC, AD, CD	ABC, ABD, BCD	1.000	0.8534	0.9511
3	9	AD, B, C	ABC, BCD	1.000	0.8492	0.8995
2	10	A, B, C, D	ABCD	1.000	0.4886	0.8295
1	11	A, B, C, D	ABCD	1.000	0.4886	0.8295

To calculate the probability of a read quorum or a write quorum, look at combinations of the links since all node availabilities are 1.0.

						R=6	W=6		R=5	W=7		R=4	W=8	
AB	BC	BD	CD			RQ	WQ		RQ	WQ		RQ	WQ	
0	0	0	0	0.0072								1		
0	0	0	1	0.0168					1			1		
0	0	1	0	0.0018		1	1	1	1			1		
0	0	1	1	0.0042		1	1	1	1	1		1	1	1
0	1	0	0	0.0648		1	1	1	1	1	1	1		
0	1	0	1	0.1512		1	1	1	1	1	1	1	1	1
0	1	1	0	0.0162		1	1	1	1	1	1	1	1	1
0	1	1	1	0.0378		1	1	1	1	1	1	1	1	1
1	0	0	0	0.0168		1	1	1	1			1		
1	0	0	1	0.0392		1	1	1	1			1		
1	0	1	0	0.0042		1	1	1	1	1	1	1	1	1
1	0	1	1	0.0098		1	1	1	1	1	1	1	1	1
1	1	0	0	0.1512		1	1	1	1	1	1	1	1	1
1	1	0	1	0.3528		1	1	1	1	1	1	1	1	1
1	1	1	0	0.0378		1	1	1	1	1	1	1	1	1
1	1	1	1	0.0882		1	1	1	1	1	1	1	1	1
					1	0.9760	0.9760	0.9760	0.9928	0.918	0.9665	1	0.8534	0.9511

					R=3	W=9		R=2	W=10		R=1	w=11
AB	BC	BD	CD		RQ	WQ		RQ	WQ		RQ	WQ
0	0	0	0	0.0072	1			1			1	
0	0	0	1	0.0168	1			1			1	
0	0	1	0	0.0018	1			1			1	
0	0	1	1	0.0042	1	1		1			1	
0	1	0	0	0.0648	1			1			1	
0	1	0	1	0.1512	1	1		1			1	
0	1	1	0	0.0162	1	1		1			1	
0	1	1	1	0.0378	1	1		1			1	
1	0	0	0	0.0168	1			1			1	
1	0	0	1	0.0392	1			1			1	
1	0	1	0	0.0042	1			1			1	
1	0	1	1	0.0098	1	1		1	1		1	1
1	1	0	0	0.1512	1	1		1			1	
1	1	0	1	0.3528	1	1		1	1		1	1
1	1	1	0	0.0378	1	1		1	1		1	1
1	1	1	1	0.0882	1	1		1	1		1	1
					0.849	1	0.899	1	0.489	0.8295	1	0.4886

The probabilities are given for the read and write quorums. Availability occurs when both a read and write quorum occurs and is weighted by the proportion of reads and writes.

21. Show how checksums can be used to detect and correct errors in a scalar by matrix multiplication for the following example. Assume a 3 x 3 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Show the corresponding column weighted matrix A_C and assume that during the mulitplication of A_C by the scalar 2 a single error has occured resulting in the following output

$$2 \cdot A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 17 & 18 \end{bmatrix}$$

The column weighted matrix A_C is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 12 & 15 & 18 \\ 37 & 44 & 51 \end{bmatrix}$$

After multiplying by the scalar 2 and with the single error, we obtain

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 17 & 18 \\ 24 & 30 & 36 \\ 74 & 88 & 102 \end{bmatrix}$$

For columns 1 and 3, S_1 and S_2 are both zero. For column 2 we calculate $S_1 = \Sigma^3_{i=1} a_{i,2}$ - WCS1 = (4 + 10 + 17) - 30 = 1, and $S_2 = \Sigma^3_{i=1} 2^{i-1} a_{i,2}$ - WCS2 = (4 + 20 + 68) - 88 = 4. Since both S_1 and S_2 are non-zero, we calculate S2/S1 = 4 = $2^{(3-1)}$ implying that $a_{3,2}$ is erroneous. We correct the error using $a_{3,2} = a_{3,2} - S_1$ = 17 - 1 = 16.