The University of Alabama in Huntsville
Electrical and Computer Engineering
Homework #5 Solution
CPE 633 01
Spring 2008

Chapter 4: Problems 5(25 points), 6(15 points), 9(15 points), 11(15 points), 14(30 points)

5. Consider an 8 x 8 butterfly network. Suppose that each processor generates a new request every cycle. This request is independent of whether or not its previous request was satisfied, and is directed to memory module 0 with probability 1/2 and to memory module i with probability 1/14, for i = 1,2,3,4,5,6,7. Obtain the expected bandwidth of this network.

   Let $q_i^{(j)}$ be the probability that output line I of stage j is idel, i.e., does not carry a memory request from a processor. Define $p_i^{(j)} = 1 - q_i^{(j)}$.

   Begin with stage 2, the input stage. The probability of a request from a processor is 1 to this stage. From the top output line of each stage-2 box, we can reach outputs 0, 1, 2, 3. The probability of a processor request to any of them is 1/2 + 1/14 +1/14 + 1/14 = 10/14 = 5/7. From the bottom output of each stage-2 box, we can reach outputs 4, 5, 6, 7. The probability of no request on this output occurs when a request from neither processor occurs and is (1 – 5/7)(1 – 5/7) = (2/7)(2/7) = 4/49. The probability of a processor request to any of them is 1/14 + 1/14 + 1/14 + 1/14 = 4/14 = 2/7. The probability of no request on this output occurs when a request from neither processor occurs and is (1 – 2/7)(1 – 2/7) = (5/7)(5/7) = 25/49. We therefore have:

   $q_i^{(2)}$ = {4/49, if i = 0, 1, 2, 3 and 25/49 if i = 4, 5, 6, 7} and
   $p_i^{(2)}$ = {45/49 = 0.918, if i = 0, 1, 2, 3 and 24/49 = 0.490 if i = 4, 5, 6, 7}.

   Now, consider stage 1. The probability of a request from a processor to this stage is $p^{(2)}$. From the top output line of the top two stage-1 boxes, we can reach outputs 0, 1. The fraction of requests going to 0, 1, 2, 3 that go to 0, 1 is (1/2 + 1/14)/(1/2 + 3/14) = (4/7)/(5/7) = 4/5. The fraction of requests going to 0, 1, 2, 3 that go to 2, 3 is (1/14 + 1/14)/(1/2 + 3/14) = (1/7)/(5/7) = 1/5. The fraction of requests going to 4, 5, 6, 7 that go to 4, 5 is (1/14 + 1/14)/(4/14) = (1/7)/(2/7) = 1/2. The fraction of requests going to 4, 5, 6, 7 that go to 6, 7 is (1/14 + 1/14)/(4/14) = (1/7)/(2/7) = 1/2.

   $q_i^{(1)} = \{(1 – 4/5p_i^{(2)})^2$, if i = 0, 1, $(1 – 1/5p_i^{(2)})^2$ if i = 2, 3 and $(1 – 1/2p_i^{(2)})^2$ if i = 4, 5, 6, 7}
   $q_i^{(1)}$ = {0.071, if i = 0, 1, 0.667 if i = 2, 3 and 0.570 if i = 4, 5, 6, 7}
   $p_i^{(1)}$ = {0.929, if i = 0, 1, 0.333 if i = 2, 3 and 0.430 if i = 4, 5, 6, 7}

   Finally, consider stage-0 boxes. If an input line to the top box is busy, it will carry a request to output 0 with probability (1/2)/(1/2 + 1/14) = (7/14)/(8/14) = 7/8. Input lines to other boxes in this stage are equally likely to request either output line.

   $q_i^{(0)} = \{(1 – 7/8p_i^{(1)})^2$, if i = 0, $(1 – 1/8p_i^{(1)})^2$ if i = 1 and $(1 – 1/2p_i^{(1)})^2$ if i = 2, 3, 4, 5, 6, 7}
   $q_0^{(0)} = 0.035$, $q_1^{(0)} = 0.781$, $q_2^{(0)} = 0.695$, $q_3^{(0)} = 0.695$, $q_4^{(0)} = 0.616$, $q_5^{(0)} = 0.616$, $q_6^{(0)} = 0.616$, $q_7^{(0)} =$ 0.616
   $p_0^{(0)} = 0.965$, $p_1^{(0)} = 0.219$, $p_2^{(0)} = 0.305$, $p_3^{(0)} = 0.305$, $p_4^{(0)} = 0.384$, $p_5^{(0)} = 0.384$, $p_6^{(0)} = 0.384$, $p_7^{(0)} = 0.384$

   The expected bandwidth is $p_0^{(0)} + p_1^{(0)} + p_2^{(0)} + p_3^{(0)} + p_4^{(0)} + p_5^{(0)} + p_6^{(0)} + p_7^{(0)}$ = 0.965 + 0.219 + 0.305 + 0.305 + 0.384 + 0.384 + 0.384 = 3.329

6. We showed how to obtain the probability, for a multistage network, that a given processor is unable to connect to any memory. In our analysis, only link failures were considered. Extend the analysis to include switchbox failures, which occur with probability qs. Assume that link and switchbox failures are all mutually independent of one another.

   For access for some point to the output end, we require that there be at least one path from that point to the output end that is free of both switchbox and link failures. We have already derived the following equations

   $\Phi(0) = (1 - q_l^2)$, $\Phi(i) = 1 - (1 - p_l\Phi(i-1))^2$

   for the case where there are no switchbox failures. If we introduce switchbox failures, we have to ensure that, for a path to exist leading to the output end of a switchbox, both the switchbox and at least one output link must be up. This leads immediately to the equations

   $\Phi(0) = p_s(1 - q_l^2)$, $\Phi(i) = 1 - p_s(1 - p_l\Phi(i-1))^2$

9. Compare the reliability of an N x M interstitial mesh (with M and N both even numbers) to that of a regular N x M mesh, given that each node has a reliability R(t) and links are fault-free. For what values of R(t) will the interstitial mesh have a higher reliability?

   For the (1, 4) scheme, the reliability is $\left(R^5(t) + 5R^4(t)(1 - R(t))\right)^{NM/4}$. The reliability of the original mesh is $R^{NM}(t)$. So, the reliability of the (1,4) scheme is greater than that of the mesh if

   $$\left(R^5(t) + 5R^4(t)(1 - R(t))\right)^{NM/4} > R^{NM}(t)\,.$$
   $$\left(R^5(t) + 5R^4(t) - 5R^5(t)\right)^{NM/4} > R^{NM}(t)$$
   $$\left(-4R^5(t) + 5R^4(t)\right)^{NM/4} > R^{NM}(t)$$
   $$\log\left(-4R^5(t) + 5R^4(t)\right)^{NM/4} > \log\left(R^{NM}(t)\right)$$
   $$\frac{NM}{4}\log\left(-4R^5(t) + 5R^4(t)\right) > NM\log\left(R(t)\right)$$
   $$\log\left(-4R^5(t) + 5R^4(t)\right) > 4\log\left(R(t)\right)$$
   $$\log\left(-4R^5(t) + 5R^4(t)\right) > \log\left(R^4(t)\right)$$
   $$\left(-4R^5(t) + 5R^4(t)\right) > R^4(t)$$
   $$-4R^5(t) > -4R^4(t)$$
   $R^5(t) < R^4(t)$ This is true for all R(t) < 1.

11. A 3 x 3 crossbar has been augmented by adding a row and a column, and input demultiplexers and output multiplexers. Assume that a switch can fail with probability q_s and when it fails all the incident links are disconnected. Also assume that all links are fault-free but multiplexers and demultiplexers can fail with probability qm. Write expressions for the reliability of the original 3 x 3 crossbar and for the fault-tolerant crossbar. (For the purposes of this question, the reliability of the fault-tolerant crossbar is the probability that there is a functioning 3 x 3 crossbar embedded within the 4 x 4 system). Will the fault-tolerant crossbar have always a higher reliability than the original 3 x 3 crossbar?

   $R_{orig} = (1 - q_s)^9$, $R_{expanded} = (1-q_m)^6[(1 - q_s)^9 + 12q_s(1 - q_s)^{15}]$

   For $R_{expanded} > R_{orig}$, $(1-q_m)^6[(1 - q_s)^9 + 12q_s(1 - q_s)^{15}] > (1 - q_s)^9$ or

$$(1-q_m)^6 > \frac{(1-q_s)9}{(1-q_s)^9 + 12(1-q_s)^{15}}$$

14. The links in an H3 hypercube are directed from the node with the lower index to the node with the higher index. Calculate the path reliability for the source node 0 and the destination node 7. Denote by $p_{i,j}$ the probability that the link from node i to node j is operational and assume that all nodes are fault-free.

The six paths from node 0 to 7 are:

$P_1 = \{x_{0,1}, x_{1,3}, x_{3,7}\}$,
$P_2 = \{x_{0,1}, x_{1,5}, x_{5,7}\}$,
$P_3 = \{x_{0,2}, x_{2,3}, x_{3,7}\}$,
$P_4 = \{x_{0,2}, x_{2,6}, x_{6,7}\}$,
$P_5 = \{x_{0,4}, x_{4,5}, x_{5,7}\}$,
$P_6 = \{x_{0,4}, x_{4,6}, x_{6,7}\}$, Finally, R = T1 + T2 + T3 + T4 + T5 + T6.

$R\{0,7\} = \text{Prob}\{P_1\} + \text{Prob}\{P_2\}\ \text{Prob}\{P_1'|P_2\} + \text{Prob}\{P_3\}\ \text{Prob}\{P_1'\cap P_2'|P_3\} + \text{Prob}\{P_4\}$
$\text{Prob}\{P_1'\cap P_2'\cap P_3'|P_4\} + \text{Prob}\{P_5\}\ \text{Prob}\{P_1'\cap P_2'\cap P_3'\cap P_4'|P_5\} + \text{Prob}\{P_6\}$
$\text{Prob}\{P_1'\cap P_2'\cap P_3'\cap P_4'\cap P_5'|P_6\}$

$\text{Prob}\{P_1\} = p_{0,1}p_{1,3}p_{3,7}$

$\text{Prob}\{P_2\} = p_{0,1}p_{1,5}p_{5,7}$
$P_{1|2} = P_1 - P_2 = \{x_{1,3}, x_{3,7}\}$
$\text{Prob}\{P_1'|P_2\} = (1 - p_{1,3}p_{3,7})$

$\text{Prob}\{P_3\} = p_{0,2}p_{2,3}p_{3,7}$
$P_{1|3} = P_1 - P_3 = \{x_{0,1}, x_{1,3}\}$, $P_{2|3} = P_2 - P_3 = \{x_{0,1}, x_{1,5}, x_{5,7}\}$,
$\text{Prob}\{P_1'\cap P_2'|P_3\} = q_{0,1} + q_{0,1}q_{1,3}(1 - p_{1,5}p_{1,7})$

$\text{Prob}\{P_4\} = p_{0,2}p_{2,3}p_{3,7}$
$P_{1|4} = P_1 - P_4 = \{x_{0,1}, x_{1,3}, x_{3,7}\}$, $P_{2|4} = P_2 - P_4 = \{x_{0,1}, x_{1,5}, x_{5,7}\}$,
$P_{3|4} = P_3 - P_4 = \{x_{2,3}, x_{3,7}\}$
$\text{Prob}\{P_1'\cap P_2'\cap P_3'|P_4\} = q_{0,1}q_{2,3} + q_{0,1}p_{2,3}q_{3,7} + p_{0,1}q_{3,7}(1 - p_{1,5}p_{5,7})$
$+ p_{0,1}p_{3,7}q_{1,3}q_{2,3}(1- p_{1,5}p_{5,7})$

$\text{Prob}\{P_5\} = p_{0,4}p_{4,5}p_{5,7}$
$P_{1|5} = P_1 - P_5 = \{x_{0,1}, x_{1,3}, x_{3,7}\}$, $P_{2|5} = P_2 - P_5 = \{x_{0,1}, x_{1,5}\}$,
$P_{3|5} = P_3 - P_5 = \{x_{0,2}, x_{2,3}, x_{3,7}\}$, $P_{4|5} = P_4 - P_5 = \{x_{0,2}, x_{2,6}, x_{6,7}\}$
$\text{Prob}\{P_1'\cap P_2'\cap P_3'\cap P_4'|P_5\} = q_{0,1}q_{0,2} + q_{0,1}p_{0,2}q_{2,3}(1 - p_{2,6}p_{6,7}) +$
$q_{0,1}p_{0,2}p_{2,3}q_{3,7}(1 - p_{2,6}p_{6,7}) + p_{0,1}q_{1,3}q_{1,5}q_{0,2} + p_{0,1}q_{1,3}q_{1,5}p_{0,2}q_{2,3}(1 - p_{2,6}p_{6,7}) +$
$p_{0,1}q_{1,3}q_{1,5}p_{0,2}p_{2,3}q_{3,7}(1 - p_{2,6}p_{6,7}) + p_{0,1}p_{1,3}q_{3,7}q_{1,5}q_{0,2} + p_{0,1}p_{1,3}q_{3,7}q_{1,5}(1 -$
$p_{2,6}p_{6,7})$

$\text{Prob}\{P_6\} = p_{0,4}p_{4,6}p_{6,7}$
$P_{1|6} = P_1 - P_6 = \{x_{0,1}, x_{1,3}, x_{3,7}\}$, $P_{2|6} = P_2 - P_6 = \{x_{0,1}, x_{1,5}, x_{5,7}\}$,
$P_{3|6} = P_3 - P_6 = \{x_{0,2}, x_{2,3}, x_{3,7}\}$, $P_{4|6} = P_4 - P_6 = \{x_{0,2}, x_{2,6}\}$,
$P_{5|6} = P_5 - P_6 = \{x_{4,5}, x_{5,7}\}$
$\text{Prob}\{P_1'\cap P_2'\cap P_3'\cap P_4'\cap P_5'|P_6\} = q_{0,1}q_{0,2}(1 - p_{4,5}p_{5,7}) + q_{0,1}p_{0,2}(1 -$
$p_{2,3}p_{3,7})q_{2,6}(1 - p_{4,5}p_{5,7}) + p_{0,1}q_{3,7}q_{5,7}(1 - p_{0,2}p_{2,6}) + p_{0,1}p_{3,7}q_{1,3}q_{0,2}q_{5,7} +$
$p_{0,1}p_{3,7}q_{1,3}q_{0,2}p_{5,7}q_{1,5}q_{4,5} + p_{0,1}p_{3,7}q_{1,3}p_{0,2}q_{2,3}q_{2,6}q_{5,7} +$
$p_{0,1}p_{3,7}q_{1,3}p_{0,2}q_{2,3}q_{2,6}p_{5,7}q_{1,5}q_{4,5}$