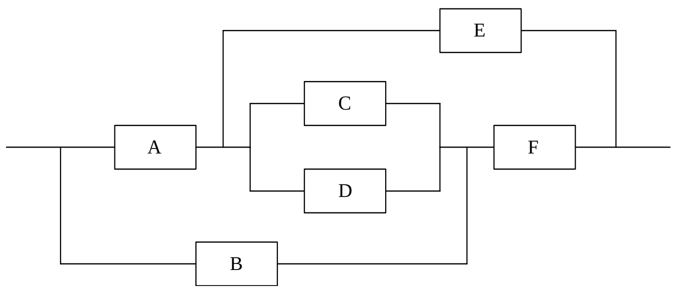
The University of Alabama in Huntsville
Electrical and Computer Engineering
CPE 633 01
Test 1 Solution
Spring 2008

1. (30 points) For the system diagram shown:
   (a) Derive the upper bound for system reliability.
   (b) Derive the lower bound for system reliability.
   (c) Derive the exact reliability formula.
   (d) If $R_A = R_B = R_C = R_D = R_E = R_F = R = e^{-\lambda t}$, find the MTTF for the system.



(a) The paths in this system are AE, ACF, ADF, and BF.

$$R_{system} \le 1 - \prod (1 - R_{pathi})$$

$R_{system} \le 1 - (1 - R_AR_E)(1 - R_AR_CR_F)(1 - R_AR_DR_F)(1 - R_BR_F)$

(b) The minimum cut sets in this system are AB, EF, BCDE, and AF.

$R_{system} \ge 1 - \prod (1 - Q_{cuti})$, where $Q_{cut\ i}$ is the probability that minimal cut set i is faulty

$R_{system} \ge 1 - [(1- (1 - R_A)(1 - R_B))(1 - (1 - R_E)(1 - R_F))\ (1 - (1 - R_A)(1 - R_F))$
$(1 - (1 - R_B)(1 - R_C)(1 - R_D)(1 - R_E))]$

(c) Expanding around F, $R_{system} = R_F(R_{system} \mid F\text{ working}) + (1 - R_F)(R_{system} \mid F\text{ not working})$

$R_{system} \mid F$ working $= (1 - (1- R_A[1 - (1 - R_E)(1 - R_C)(1 - R_D)])(1 - R_B))$
$= (1 - (1- R_A[1 - (1 - R_E - R_C + R_CR_E)(1 - R_D)])(1 - R_B))$
$= (1 - (1- R_A[1 - (1 - R_E - R_C + R_CR_E - R_D + R_DR_E + R_CR_D - R_CR_DR_E)])(1 - R_B))$
$= (1 - (1- R_A[R_E + R_C - R_CR_E + R_D - R_DR_E - R_CR_D + R_CR_DR_E)])(1 - R_B))$
$= (1 - (1- R_AR_E - R_AR_C + R_AR_CR_E - R_AR_D + R_AR_DR_E + R_AR_CR_D - R_AR_CR_DR_E)(1 - R_B))$
$= (1 - (1- R_AR_E - R_AR_C + R_AR_CR_E - R_AR_D + R_AR_DR_E + R_AR_CR_D - R_AR_CR_DR_E - R_B$
$+ R_AR_BR_E + R_AR_BR_C - R_AR_BR_CR_E + R_AR_BR_D - R_AR_BR_DR_E - R_AR_BR_CR_D + R_AR_BR_CR_DR_E))$
$= R_AR_E + R_AR_C - R_AR_CR_E + R_AR_D - R_AR_DR_E - R_AR_CR_D + R_AR_CR_DR_E + R_B$
$- R_AR_BR_E - R_AR_BR_C + R_AR_BR_CR_E - R_AR_BR_D + R_AR_BR_DR_E + R_AR_BR_CR_D - R_AR_BR_CR_DR_E$

$R_{system} \mid F$ not working $= R_AR_E$

$R_{system} = R_F(R_AR_E + R_AR_C - R_AR_CR_E + R_AR_D - R_AR_DR_E - R_AR_CR_D + R_AR_CR_DR_E + R_B - R_AR_BR_E$
$- R_AR_BR_C + R_AR_BR_CR_E - R_AR_BR_D + R_AR_BR_DR_E + R_AR_BR_CR_D - R_AR_BR_CR_DR_E) + (1 -$
$R_F)R_AR_E$

(d) $R_{system} = -R^6 + 4R^5 - 6R^4 + 2R^3 + 2R^2$

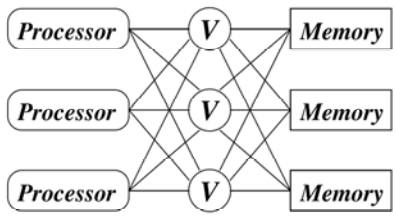
$$MTTFsystem = \int_0^\infty Rsystem(t)dt = \int_0^\infty -e^{-6\lambda t} + 4e^{-5\lambda t} - 6e^{-4\lambda t} + 2e^{-3\lambda t} + 2e^{-2\lambda t}$$

$$= \frac{-e^{-6\lambda t}}{-6\lambda} + \frac{4e^{-5\lambda t}}{-5\lambda} + \frac{-6e^{-4\lambda t}}{-4\lambda} + \frac{2e^{-3\lambda t}}{-3\lambda} + \frac{2e^{-2\lambda t}}{-2\lambda}\Bigg|_0^\infty$$

$$= [0-0+0-0-0] - \left[\frac{1}{6\lambda} - \frac{4}{5\lambda} + \frac{3}{2\lambda} - \frac{2}{3\lambda} - \frac{1}{\lambda}\right]$$

$$= \frac{-1}{6\lambda} + \frac{4}{5\lambda} - \frac{3}{2\lambda} + \frac{2}{3\lambda} + \frac{1}{\lambda} = \frac{-5+24-45+20+30}{30\lambda} = \frac{24}{30\lambda} = \frac{4}{5\lambda}$$

2. (20 points) Consider the processor/memory configuration shown below. List the conditions under which it will fail,and compare them to a straightforward TMR configuration in which each unit consists of a processor and a memory. Denote by Rp, Rm , and Rv, the reliability of a processor, a memory, and a voter, respectively, and write expressions for the reliability of the two TMR configurations.



The conditions under which this configuration would fail are

2 or 3 processors fail, 2 or 3 memories fail, 2 or 3 voters fail

Each of these units follows the TMR

$$R_{2\_of\_3} = \sum_{i=2}^{3} \binom{3}{i} R^i(t)(1-R(t))^{3-i} = \binom{3}{2} R^2(t)\ (1-R(t))^{3-2} + \binom{3}{3} R^3(t)(1-R(t))^{3-3}$$

$$= \frac{3!}{2!1!}\left(R^2(t)(1-R(t))\right) + \frac{3!}{3!0!} R^3(t) = 3\left(R^2(t)(1-R(t))\right) + R^3(t) = -2R^3(t) + 3R^2(t)$$

$$R_{system} = \left[-2R_p{}^3(t) + 3R_p{}^2(t)\right]\left[-2R_m{}^3(t) + 3R_m{}^2(t)\right]\left[-2R_v{}^3(t) + 3R_v{}^2(t)\right]$$

With each unit having a processor and memory and having one voter.

$$R_{system} = R_v\left[-2\left(R_p(t)R_m(t)\right)^3 + 3\left(R_p(t)R_m(t)\right)^2(t)\right]$$

3. (15 points) Derive all codewords for the separable 6-bit cyclic code based on the generating polynomial $X^3 + X^2 + 1$.

```
            1
1101  001000
        1101
         101
```

```
           11
1101  010000
        1101
         1010
         1101
          111
```

```
           10
1101  011000
        1101
         010
```

```
          1110
1101  1000000
      1101
       1010
       1101
        1110
        1101
         010
```

```
          110
1101  101000
      1101
       1110
       1101
        110
```

```
          100
1101  110000
      1101
         100
```

```
           11
1101  111000
      1101
        1100
        1101
         001
```

Codewords: `000000, 001101, 010111, 011010, 100010, 101110, 110100, 111001`

4. (20 points) A communication channel has a probability of 10^{-4} that a bit transmitted is erroneous. The data rate is 6000 bits per second (bps). Data packets contain 148 information bits, a 16-bit CRC for error detection, and 0, 8, or 16 bits for error correction coding (ECC). Assume that if 8 ECC bits are added all single bit errors can be corrected, and if 16 ECC bits are added all double bit errors can be corrected.
   (a) Find the throughput in information bits per second of a scheme consisting of error detection with retransmission of bad packets (i.e., no error correction).
   (b) Find the throughput if 8 ECC check bits are used, so that single bit errors can be corrected. Uncorrectable packets must be retransmitted.
   (c) Finally find the throughput if 16 ECC check bits are appended, so that two bit errors can be corrected. As in (b), uncorrectable packets must be retransmitted. Would you recommend increasing the number of ECC check bits from 8 to 16?

   (a) Each packet contains 164 bits. If any error occurs, it is detected (assuming that the CRC always works) and the packet is discarded. The probability that a packet has no errors is $(164!/(164!0!))(1 - 10^{-4})^{164} = 0.9837$. The data rate of the code is 148/164 = 0.9024. Thus, the throughput in bits per second is 0.9837 *0.9024 * 6000 = 5326
   (b) With the addition of 8 ECC check bits, each packet contains 172 bits. The probability that a packet has at most one error is $(172!(172!0!))(1 - 10^{-4})^{172} + (172!/(171!1!)) * 10^{-4}(1 - 10^{-4})^{171} = 0.9994$. The rate of the code is now 148/172 = 0.8430. Thus, the throughput with single bit error correction is 0.9994 * 0.843 * 6000 = 5055.
   (c) The second ECC byte increases the packet size to 180. The probability that a packet of 180 bits has no more than two errors is $(180!(180!0!))(1 - 10^{-4})^{180} + (180!/(179!1!))(10^{-4}(1 - 10^{-4})^{179} + (180!(178!2!))(10^{-4})^{2}(1 - 10^{-4})^{178} = 0..$ The code rate is now 148/180 = 0.8222, so the throughput is 0.9999 * 0.8222 * 6000 = 4933. Increasing the error correction capability in this case resulted in a reduction in the throughput, so no.

5. (15 points) Consider two computers A and B. (a) Assuming an exponential distribution, what is the probability that at least one will survive 10,000 hours if their failure rate is 1 failure per million hours? (b) What is the probability that both will survive 10,000 hours? (c) What is the probability that A will survive 15,000 hours if it survived 5,000 hours?

   (a)
$$
\begin{aligned}
&P\{T_A > 10{,}000 \vee T_B > 10{,}000\} = 1 - P\{T_A < 10{,}000 \cap T_B < 10{,}000\}\\
&= 1 - P\{T_A < 10{,}000 \cap T_B < 10{,}000\} = 1 - (FA(10000)FB(10000))\\
&= 1 - (1 - R_A(10000))(1 - R_B(10000)) = 1 - (1 - 2R(10000) + R^2(10000))\\
&= 2R(10000) - R^2(10000)) = 2e^{-10^{-6}(10000)} - e^{-2*10^{-6}(10000)} = 0.9999
\end{aligned}
$$

   (b)
$$
\begin{aligned}
&P\{T_A > 10{,}000 \cap T_B > 10{,}000\} = P\{T_A > 10{,}000\}P\{T_A > 10{,}000\}\\
&= R_A(10{,}000)R_B(10{,}000) = e^{-10^{-6}(10000)}e^{-10^{-6}(10000)} = 0.9802
\end{aligned}
$$

   (c)
$$
P\{T_A > 15{,}000 \mid T_A > 5000\} = \frac{P\{T_A > 15{,}000 \cap T_A > 5000\}}{P\{T_A > 5000\}} = \frac{P\{T_A > 15000\}}{R_A(5000)}
$$

$$
= \frac{R_A(15000)}{e^{-10^{-6}(5000)}} = \frac{e^{-10^{-6}(15000)}}{e^{-10^{-6}(5000)}} = 0.9048
$$